College Algebra

Version 2

A Partnership between Institutions in the Utah System of Higher Education

Salt Lake Community College

University of Utah

Weber State University

Acknowledgements

The development of this OER textbook was initiated by Salt Lake Community College to provide its students with a low-cost textbook. University of Utah and Weber State University later joined the project. Collectively, a rigorous text has been developed by these institutions. The body of the text conforms to the College Algebra Learning Outcomes identified by the Utah System of Higher Education.

Salt Lake Community College faculty began the project, with Ruth Trygstad taking the lead and Spencer Bartholomew joining her. Peter Trapa, seeing the need for a unified approach to College Algebra, brought the University of Utah into the project with Maggie Cummings representing their perspective. Weber State University, represented by Afshin Ghoreishi, also joined the project. Rounding out the writing group was Sarah Nicholson, a concurrent enrollment instructor from Kearns High School, Granite School District.

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Content is largely adapted from *College Algebra, 3rd Corrected Edition*, by Carl Stitz and Jeff Zeager. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. We appreciate the time and effort they invested in creating and sharing their textbook. You will note there are many places where we have diverged significantly from their original text. We hope our perspective contributes beneficially to the world of OER mathematics material. Additional content, primarily examples and exercises, are from *Algebra and Trigonometry*, by Jay Abramson, et al. It is licensed under a Creative Commons Attribution 4.0 International License and can be found at https://openstax.org.

This second version of the textbook is the result of many hours of collaborative revisions, reviews, and meetings beginning in 2020 with completion in Summer 2021. The textbook revision process has been led by Ruth Trygstad and Afshin

Ghoreishi, along with team members Michael van Opstall (University of Utah), Spencer Bartholomew, Sarah (Nicholson) Featherstone, Kyle Costello (SLCC), and Piotr Runge (SLCC). Jie Gu (SLCC) has coordinated the alignment and creation of accompanying online homework exercises. Additionally, Rebecca Noonan Heale (University of Utah) and Shawna Haider (SLCC) have developed the online homework that aligns with this textbook. Camille Diaz (SLCC) has made significant contributions to this revised version as well as the original version of the texbook.

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CHAPTER 1
GETTING STARTED WITH FUNCTIONS

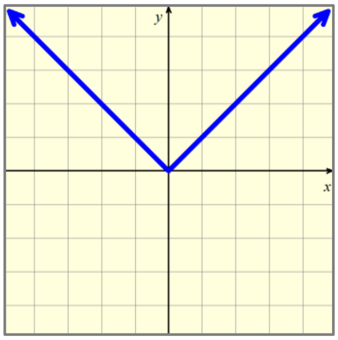


Figure 1.0. 1

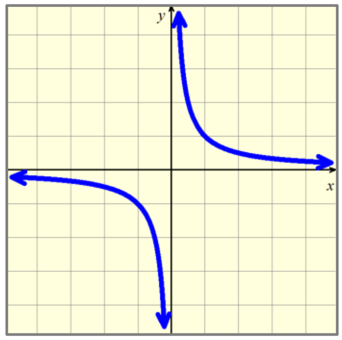


Figure 1.0. 2

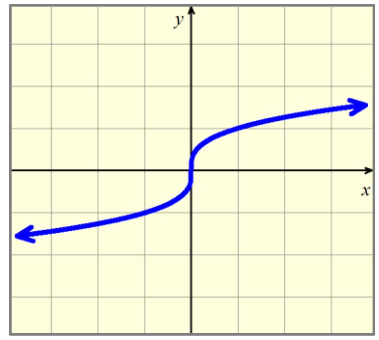


Figure 1.0. 3

Chapter Outline

Introduction

In Chapter 1 we build on understandings of functions from Elementary Algebra. The goal throughout the chapter is to help you a) gain fluidity and flexibility among the various representations of functions (verbal, analytic, numeric, and graphic) and b) understand how the various representations are related to one another and might be used in understanding problem situations more thoroughly. By the end of the chapter you should have a firm grasp on how to determine the domain and range of a variety of functions, analytically or from a graph. You will understand how and why changes to the formula of a function, such as changing $f(x)=\sqrt{x}$ to $f(x)=\sqrt{x}-a$ or $f(x)=\sqrt{x-a}$, affect the domain and range and thus the graphical representation of the function. We will build on these ideas in future chapters.

Section 1.1 deals with functions primarily from an analytic perspective. We start by reviewing from Elementary/Intermediate Algebra the definition of a function and how to determine if a relation expressed as a set of ordered pairs, an equation, or a graph represents a function. Definitions for domain and range are also introduced here and you will learn how to determine both domain and range from a graph, a skill that will be enormously important throughout the entire course. A key idea about domain you should develop in this first section revolves around the notion of 'input restrictions.' You will note that there are no restrictions for inputs to polynomial functions, thus the domain of a polynomial is always all real numbers. However, from your previous work with rational expressions, you know that one cannot divide by zero; hence, the restriction(s) for a rational function will always involve any value that results in a zero in the denominator. Further, you have learned one cannot take the square root (or any even root) of a negative number, if the output is to be a real number. Thus, for even-root functions, we restrict the input to values that give us arguments that are greater than or equal to zero.

In Section 1.2 we explore the link between analytic and graphic representations of functions. We start with the introduction of several key ‘parent’ or ‘toolkit’ functions (these terms are used interchangeably throughout the text) and explore them numerically, analytically, and graphically, with a strong focus on what the domain and range of each are, and how the domain and range are evident in the different representations. We also explore piecewise-defined functions, what it means for a function to be odd/even or have symmetry, where functions are increasing/decreasing, and maximum and minimum values of functions, all both graphically and analytically, as appropriate.

Section 1.3 deals with transformations of toolkit functions and how the transformations affect the domain and range numerically, analytically, and graphically. The primary goal is to understand transformations as more than a set of rules, e.g. to understand why changes to the argument of a function may affect the domain and thus result in a horizontal change of the graph; whereas a change to the function value may affect the range of the function and thus its graph vertically. By the end of the section, you should be able to graph a variety of transformations of parent graphs and state their domains and ranges.

Section 1.4 deals with operations with functions, namely addition, subtraction, multiplication, division, and composition. A great deal of attention is paid to composing functions and finding the domain of compositions.

Section 1.5 deals with inverse functions: what they are, how to find them, and why they are useful. You will explore all this numerically, analytically, and graphically. While you may have worked with inverses in previous courses, we caution you to think carefully about this section as ideas presented here are fundamental to understanding the relationship between logarithms and exponentials that will be explored later in the course. Throughout the section, special attention should be paid to how to find an inverse in each of the representations of a function: numerically by switching the input and output, analytically by expressing the input variable in terms of the output variable, and graphically by reflecting the graph of the function across the line $y = x$. Each of these methods sheds light on the meaning of the inverse. Lastly, you will learn what it means for a function to be one-to-one and how that information may be useful.

1.1 Introduction to Functions

Learning Objectives

- Determine whether a relation represents a function.
- Use the vertical line test to identify graphs of functions.
- Find the domain and range from the graph of a function.
- Find input and output values of a function.
- Find the domain from the equation of a function.

One of the core concepts in College Algebra is the function. We will define a function as a special kind of relation. Thus, we initiate our study of functions by discussing relations.

Relations

We begin with the definition of a **relation**.

Definition 1.1. A **relation** is a set of ordered pairs. The set of first components of the ordered pairs is called the **domain** and the set of second components of the ordered pairs is called the **range**.

Consider the following set of ordered pairs: $\{(1,2), (2,4), (3,6), (4,8), (5,10)\}$. In this relation, the domain is $\{1, 2, 3, 4, 5\}$ and the range is $\{2, 4, 6, 8, 10\}$.

Each element in the domain is known as an **input** and corresponds to at least one element in the range; that corresponding element in the range is known as an **output**. An equation in two variables also represents a relation; the ordered pairs are the corresponding input and output values that satisfy the given equation. Of the two variables, the **independent variable** represents input values and the **dependent variable** represents output values. Unless otherwise stated, for an equation in the two variables x and y, x is the independent variable and y is the dependent variable. We also note that any graph in the plane represents a relation since each point on a graph is an ordered pair. Ordered pairs may also be represented in a table, as follows:

x	1	2	3	4	5
y	2	4	6	8	10

Functions

As mentioned at the beginning of this section, a **function** is a special kind of relation. To be considered a function, each element in the domain of a relation must be paired with exactly one element in the range. Equivalently, any two ordered pairs having the same first component must also have the same second component.

Definition 1.2. A **function** is a relation in which any two ordered pairs with the same first component also have the same second component.[1]

We note that the relation $\{(-1,2), (0,2), (7,-6), (-4,-5), (-5,3)\}$ is a function since no two ordered pairs have the same first component. Likewise, $\{(1,\text{odd}), (2,\text{even}), (3,\text{odd}), (4,\text{even}), (5,\text{odd})\}$, the set of ordered pairs that relates the first five natural numbers to the words 'even' and 'odd', is a function since each ordered pair has a unique first component.

For an example of a relation that is not a function, consider the set of ordered pairs $\{(\text{odd},1), (\text{even},2), (\text{odd},3), (\text{even},4), (\text{odd},5)\}$ that relates the words 'even' and 'odd' to the first five natural numbers. Here, the two ordered pairs $(\text{odd},1)$ and $(\text{odd},3)$ have the same first component but have different second components. This violates the definition of a function, so the relation is not a function.

The next three examples include relations defined by sets of ordered pairs, equations, and graphs. We will use **Definition 1.2** to determine if these relations are also functions.

Example 1.1.1. Determine if the following relations, represented by sets of ordered pairs, are functions.

1. $R_1 = \{(-2,1), (1,3), (1,4), (3,-1)\}$
2. $R_2 = \{(-2,1), (1,3), (2,3), (3,-1)\}$

Solution.

1. A quick scan of the ordered pairs in R_1 reveals that two ordered pairs have a first component of 1, and that the first component of 1 is matched with two different second components, namely 3 and 4. Hence, R_1 is not a function.

[1] You will see other definitions for functions throughout your study of mathematics. It is worth noting that these definitions are simply different, but equivalent, ways of identifying this special type of relation.

2. Every first component in R_2 occurs only once. Thus, R_2 is a function.

□

In the previous example, the relation R_2 contained two ordered pairs with the same second component, namely $(1,3)$ and $(2,3)$. We note that, in order to say that R_2 is a function, we just need to ensure the same first component is not used with more than one second component. We can similarly test an equation where x and y represent ordered pairs, with x being the independent variable and y being the dependent variable.

Example 1.1.2. Determine if the following relations, expressed as equations, are functions.

1. $y = 5x - 2$
2. $x = y^2 + 1$
3. $x^2 + y^3 = 1$

Solution.

1. For the relation $y = 5x - 2$, the first component of each ordered pair is represented by the variable x and the second component is represented by y. Since, for each value of x, we get a unique value for y, any two ordered pairs with the same first component will have the same second component. Thus, $y = 5x - 2$ is a function.
2. The relation $x = y^2 + 1$, with the first component represented by x and the second component represented by y, has ordered pairs $(2,1)$ and $(2,-1)$. Since the first component of $x = 2$ corresponds to two different second components, namely $y = 1$ and $y = -1$, the relation $x = y^2 + 1$ is not a function.
3. Here, we solve the equation $x^2 + y^3 = 1$ for y to determine whether each choice of x results in a single corresponding value for y.

$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

For any x-value we choose, the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** real value of y. Hence, this equation represents a function.

□

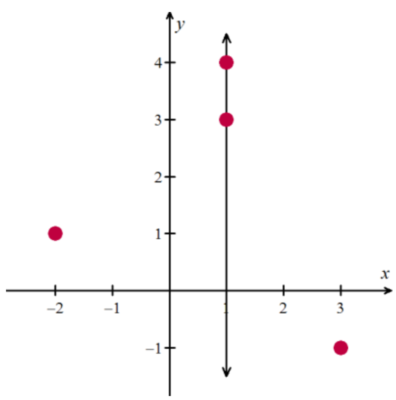
We note that for ordered pairs comprising the graph of the function $y = 5x - 2$, each x-value is associated with a single y-value. Since x is the independent variable and y is the dependent variable, we say the equation represents ***y* as a function of *x***. Similarly, we can say $x^2 + y^3 = 1$ represents y as a function of x.

We next introduce the vertical line test, which we will use to determine if a graph represents a function.

The Vertical Line Test

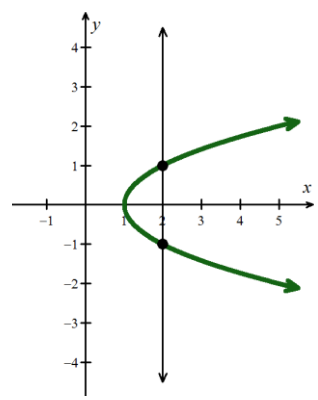
To see what the function concept means geometrically, we graph $R_1 = \{(-2,1), (1,3), (1,4), (3,-1)\}$ and $x = y^2 + 1$. These are the relations we determined were not functions in the previous two examples.

Figure 1.1. 1



Graph of R_1 with vertical line $x = 1$

Figure 1.1. 2



Graph of $x = y^2 + 1$ with vertical line $x = 2$

We note that the vertical line $x = 1$ intersects the graph of R_1 at two points, $(1,3)$ and $(1,4)$. From the graph of $x = y^2 + 1$, we see that the vertical line $x = 2$ intersects the graph at the two points $(2,1)$ and $(2,-1)$. Contemplating on these two examples, it is evident that a vertical line intersecting a graph more than once shows, in particular, that two points with the same *x*-coordinate have different *y*-coordinates, and that the graph does not represent a function.

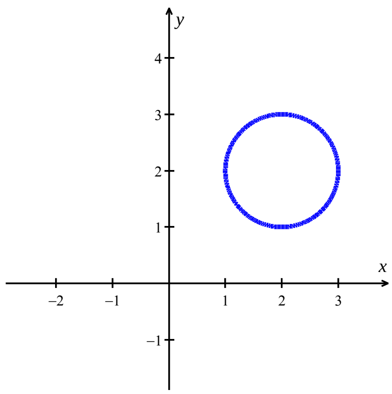
What if no vertical line intersects a graph more than once? Then there would **not** be two different points with the same *x*-coordinate, and the graph **would** represent a function. The vertical line test follows from these observations.

> **Theorem 1.1. The Vertical Line Test:** A graph represents a function if no vertical line intersects it at more than one point.

It is worth taking some time to meditate on the vertical line test, checking our understanding of the concept of a function and the concept of a graph.

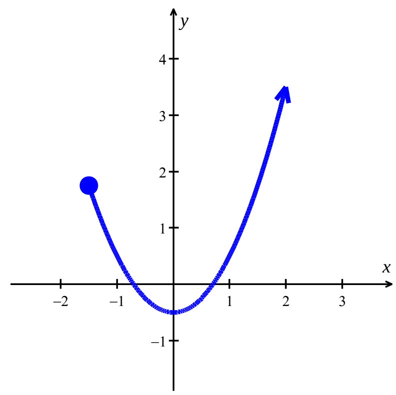
Example 1.1.3. Use the vertical line test to determine which of the following relations represents y as a function of x.

Figure 1.1. 3



The graph of relation S

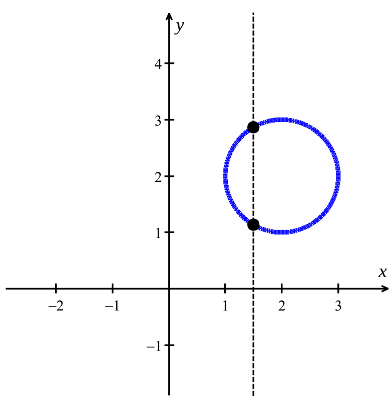
Figure 1.1. 4



The graph of relation T

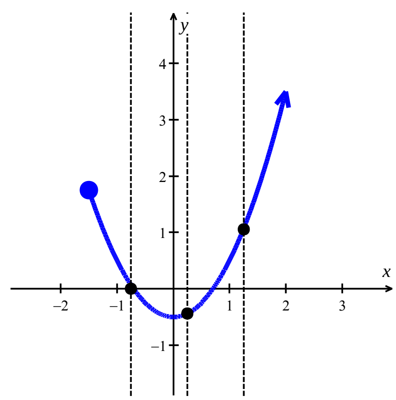
Solution. Looking at the graph of S, we can easily imagine a vertical line crossing (intersecting) the graph more than once. Hence, S is not a function. In the graph of T, every vertical line crosses (intersects) the graph at most once, so T is a function.

Figure 1.1. 5



Vertical line intersecting the graph of S twice

Figure 1.1. 6



Vertical lines intersecting the graph of T once

□

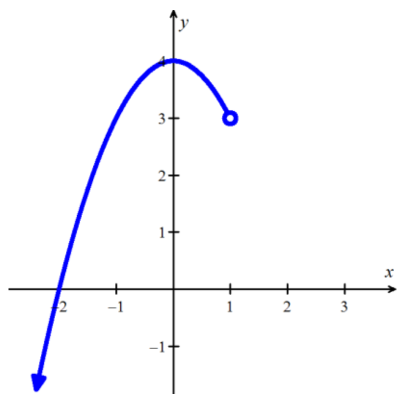
In the previous example, we say the graph of the relation S **fails** the vertical line test whereas the graph of T **passes** the vertical line test. Note that in the graph of S there are infinitely many vertical lines that intersect the graph more than once. However, to fail the vertical line test, all that is needed is one vertical line that intersects the graph at two points, as was evidenced in the graph of the relation R_1, shown prior to **Example 1.1.3**.

Determining the Domain and Range from a Graph

We next identify the domain and range of functions represented by graphs. Because the domain refers to the set of first components of the ordered pairs, where the ordered pairs are coordinates of points on the graph, the domain consists of all possible *x*-values[2] of points on the graph. The range, similarly, is the set of second components and consists of all possible *y*-values[3] of points on the graph.

Example 1.1.4. Find the domain and range of the function G whose graph is shown below.

Figure 1.1. 7



The graph of G

Solution. To find the domain and range of G, we determine which *x*- and *y*-values occur as coordinates of points on the given graph. Before going further, we need to pay attention to two subtle notations on the graph. The arrowhead on the lower left corner of the graph indicates that the graph continues to curve downward to the left forevermore. The open circle at $(1,3)$ indicates that the point $(1,3)$ is not on the graph, but that all points on the curve leading up to that point are on the graph.

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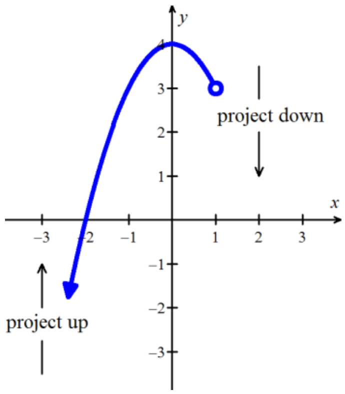
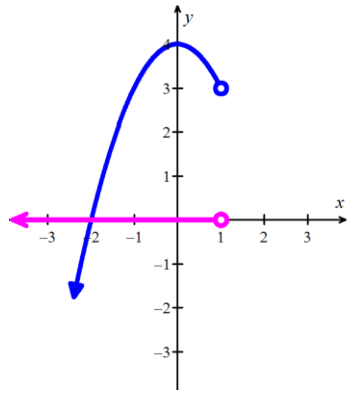
[2] Assuming *x* is the independent variable.
[3] Assuming *y* is the dependent variable.

Figure 1.1. 8



The graph of G

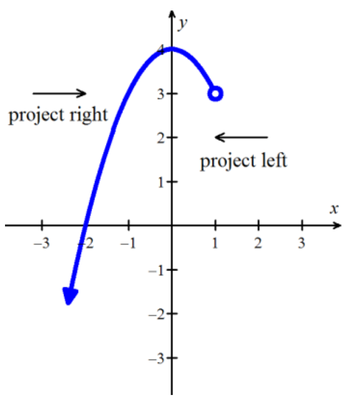
Figure 1.1. 9



G and its projection to the *x*-axis

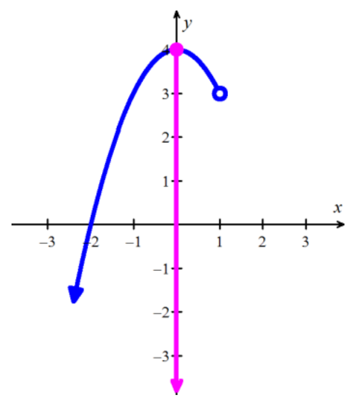
To find the domain, it may be helpful to imagine collapsing the curve vertically to the *x*-axis and determining the portion of the *x*-axis that gets covered. This is called **projecting** the curve to the *x*-axis. We see from the figure that if we project the graph of G to the *x*-axis, we get all real numbers less than 1. Using interval notation, we write the domain of G as $(-\infty,1)$. To determine the range of G, we project the curve to the *y*-axis as follows.

Figure 1.1. 10



The graph of G

Figure 1.1. 11



G and its projection to the *y*-axis

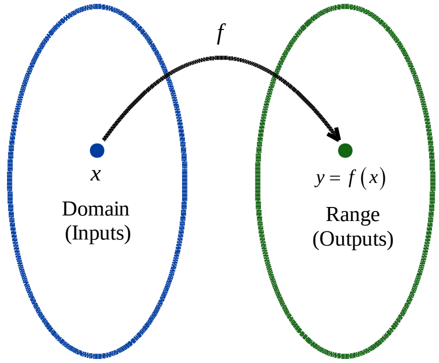
Note that even though there is an open circle at $(1,3)$, we still include the *y*-value of 3 in our range since the point $(-1,3)$ is on the graph of G. We see that the range of G is all real numbers less than or equal to 4 or, in interval notation, $(-\infty,4]$.

□

Function Notation

In **Definition 1.2**, we described a function as a special kind of relation, one in which any two ordered pairs with the same first component also have the same second component. Since we also refer to first components as **inputs** and second components as **outputs**, we can think of a function f as a process by which each input is matched with only one output.

Figure 1.1. 12



Additionally, since the independent variable x represents input values and the dependent variable y represents output values, we can think of a function f as a process by which each input value of x is matched with a unique output value of y. Since the output is completely determined by the input x and the process f, we symbolize the output with the **function notation** $f(x)$, which is read as 'f of x'. In other words, $f(x)$ is the output that results from applying the process f to the input x. In this case, the parentheses do not indicate multiplication as they often do elsewhere in algebra.

Finding Input and Output Function Values

Evaluating formulas using function notation is a key skill for success in this and many other mathematics courses. In the following example, we evaluate output values for the given function, and determine the input value(s) required to result in a given output.

Example 1.1.5. Let $f(x) = -x^2 + 3x + 4$.

1. Find and simplify the following.
   (a) $f(-1)$, $f(0)$, $f(2)$
   (b) $f(a)$
2. Solve $f(x) = -14$.

Solution.

1. (a) To find $f(-1)$ when $f(x) = -x^2 + 3x + 4$, we replace every occurrence of x in $f(x)$ with -1.

$$\begin{aligned} f(x) &= -x^2 + 3x + 4 \\ f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly, we find $f(0)$ and $f(2)$.

$$\begin{aligned} f(0) &= -(0)^2 + 3(0) + 4 \\ &= 4 \end{aligned} \qquad\qquad \begin{aligned} f(2) &= -(2)^2 + 3(2) + 4 \\ &= -4 + 6 + 4 \\ &= 6 \end{aligned}$$

(b) To find $f(a)$, we replace every occurrence of x with the quantity a.

$$\begin{aligned} f(a) &= -(a)^2 + 3(a) + 4 \\ &= -a^2 + 3a + 4 \end{aligned}$$

2. Since $f(x) = -x^2 + 3x + 4$, the equation $f(x) = -14$ is equivalent to $-x^2 + 3x + 4 = -14$. We have

$$\begin{aligned} -x^2 + 3x + 4 &= -14 \\ -x^2 + 3x + 18 &= 0 \\ x^2 - 3x - 18 &= 0 \qquad \text{after multiplying through by } -1 \\ (x+3)(x-6) &= 0 \end{aligned}$$

Setting each factor equal to zero results[4] in the solutions $x = -3$ or $x = 6$, which can be verified by checking that $f(-3) = -14$ and $f(6) = -14$.

□

In **Example 1.1.5**, note the practice of using parentheses when substituting algebraic values into functions. This practice is highly recommended, as it will reduce careless errors.

Determining the Domain of a Function from a Formula

Before proceeding with finding the domain of a function defined by a formula, we consider the following definition.

Definition 1.3. The **domain** of a function is the set of all input values for which the function is defined.

To determine the domain of the function $r(x) = \dfrac{2x}{x^2 - 9}$, we look for the set of all x-values for which the function is defined. At issue are those values of x that result in r having a denominator of zero, since

[4] Our assumption is that you can solve quadratic equations by factoring or applying the Quadratic Formula. A brief review is included in **Section 2.1**, as a foundation for further exploration of quadratic functions.

division by zero is not allowed.[5] We determine which x-values result in division by zero by setting the denominator equal to zero and solving:

$$x^2 - 9 = 0$$
$$(x-3)(x+3) = 0$$

Setting each factor equal to zero results in $x = 3$ and $x = -3$. We note that, as long as we substitute numbers other than 3 and -3 for x, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation as $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

When a formula for a function is given, we assume that the function is valid for all real numbers that make arithmetic sense when substituted into the formula. This set of numbers is also called the **implicit domain**[6] of the function. At this stage, there are only two mathematical transgressions we need to avoid: division by zero and extraction of even roots from negative numbers. The following example illustrates these concepts.

Example 1.1.6. Find the domain of the following functions.

1. $f(x) = 2x^3 - x + 3$
2. $g(x) = \dfrac{x^2 - 2x + 3}{2}$
3. $h(x) = \dfrac{5}{x+3}$
4. $I(x) = \sqrt{2-x}$
5. $r(t) = \sqrt[3]{t-2}$
6. $J(x) = \dfrac{x-3}{x^2 + 2x - 3}$

Solution.

1. The function $f(x) = 2x^3 - x + 3$ requires that the number x be cubed and then multiplied by 2. From the result of these two operations, the number x is subtracted and then the number 3 is added. These operations can be performed on any real number x and so the domain of f is all real numbers, or $(-\infty,\infty)$. We note that f is a polynomial function,[7] and through similar reasoning can conclude that the domain of any polynomial function is all real numbers.

2. The function $g(x) = \dfrac{x^2 - 2x + 3}{2}$ can be rewritten as $g(x) = \dfrac{1}{2}x^2 - x + \dfrac{3}{2}$, which we recognize as a polynomial function. Our conclusion from part 1, that the domain of a polynomial function is all real numbers, applies here. We identify the domain of g as being $(-\infty,\infty)$.

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[5] Take a moment to contemplate why division by zero is undefined, researching if necessary.

[6] An **explicit domain** is a domain that is specifically stated, such as $f(x) = x - 1$ for $x \geq 2$.

[7] You have seen polynomials in prior mathematics classes. A formal introduction to polynomial functions is included in **Section 2.2**.

3. In finding the domain of $h(x)=\dfrac{5}{x+3}$, we notice that we have a denominator containing x. Any values of x that would result in division by zero must be excluded from the domain. To find those values, we set the denominator equal to 0. We get $x+3=0$, or $x=-3$. In order for a real number x to be in the domain of h, $x\neq -3$. In interval notation, the domain is $(-\infty,-3)\cup(-3,\infty)$.

4. The potential disaster for $I(x)=\sqrt{2-x}$ occurs when the radicand is negative. To avoid this, we set $2-x\geq 0$ and solve for x:

$$\begin{aligned} 2-x&\geq 0\\ -x&\geq -2\\ x&\leq 2 \end{aligned}$$

What this shows is that as long as $x\leq 2$, then $2-x\geq 0$ and the value of $I(x)$ is a real number. Thus, the domain is $(-\infty,2]$.

5. The function $r(t)=\sqrt[3]{t-2}$ is hauntingly close to $I(x)$, with one key difference. Whereas the expression for $I(x)$ includes an even indexed root (namely square root), the formula for $r(t)$ involves an odd indexed root (the cube root). Since odd roots of real numbers (including negative real numbers) are real numbers, there is no restriction on the inputs to r. Hence, our domain is $(-\infty,\infty)$.

6. Once again, $J(x)=\dfrac{x-3}{x^2+2x-3}$ has a potential issue with values of x resulting in a denominator of zero. To determine those values, we set the denominator equal to zero and solve for x.

$$\begin{aligned} x^2+2x-3&=0\\ (x+3)(x-1)&=0 \end{aligned}$$

We find that $x=-3$ and $x=1$ result in a denominator of zero, so both values must be excluded from the domain. While it is tempting to do something with the numerator of $x-3$, there are no values that must be excluded from the numerator and so, after excluding $x=-3$ and $x=1$, the domain is $(-\infty,-3)\cup(-3,1)\cup(1,\infty)$.

□

It is worth noting the importance of finding the domain of a function before simplifying. As an example, although the function $K(x)=\dfrac{3x^2}{x}$ simplifies to $3x$, the domain excludes $x=0$. It would be inaccurate to write $K(x)=3x$ without adding the stipulation that $x\neq 0$.

1.1 Exercises

1. What is the difference between a relation and a function?

2. Why does the vertical line test tell us whether the graph of a relation represents a function?

In Exercises 3 – 8, determine whether or not the relation represents a function. Find the domain and range of those relations that are functions.

3. $\{(-3,9), (-2,4), (-1,1), (0,0), (1,1), (2,4), (3,9)\}$

4. $\{(-3,0), (1,6), (2,-3), (4,2), (-5,6), (4,-9), (6,2)\}$

5. $\{(-3,0), (-7,6), (5,5), (6,4), (4,9), (3,0)\}$

6.

x	y
-5	-4
1	2
4	5
8	7
11	16

7.

x	y
0	-2
3	2
3	6
8	7
12	16

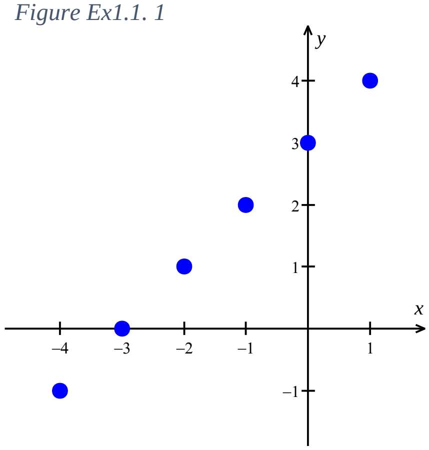
8.

x	y
-5	-1
1	3
6	3
8	9
16	10

In Exercises 9 – 30, determine if the relation represents y as a function of x. Find the domain and range of those relations that are functions.

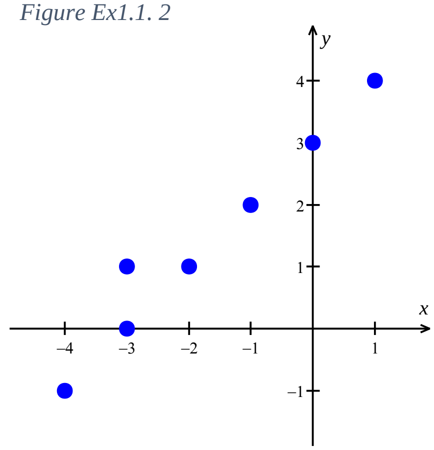
9.

Figure Ex1.1. 1



10.

Figure Ex1.1. 2



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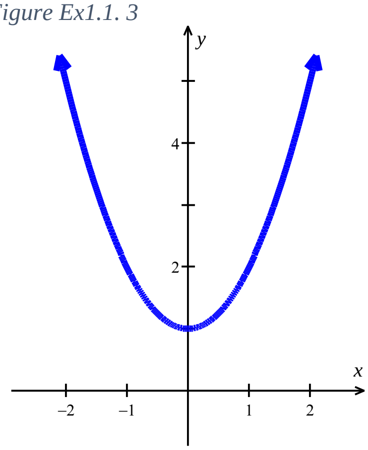

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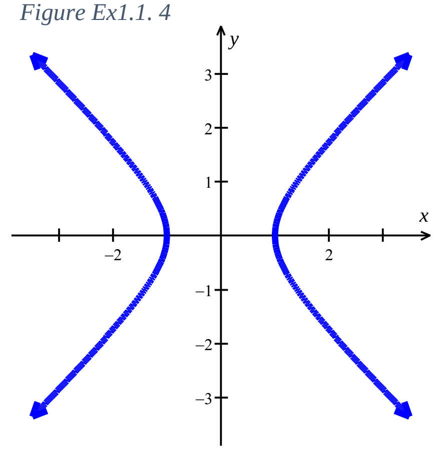

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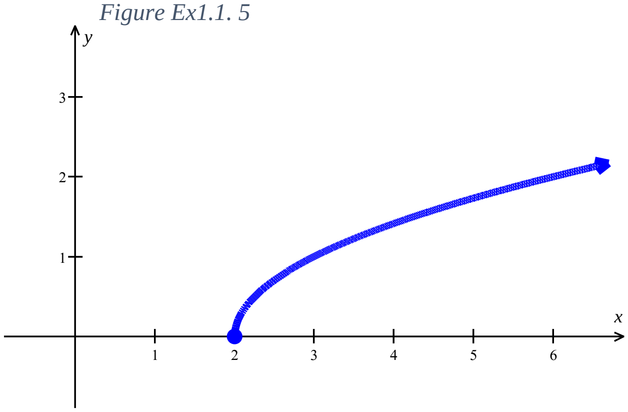

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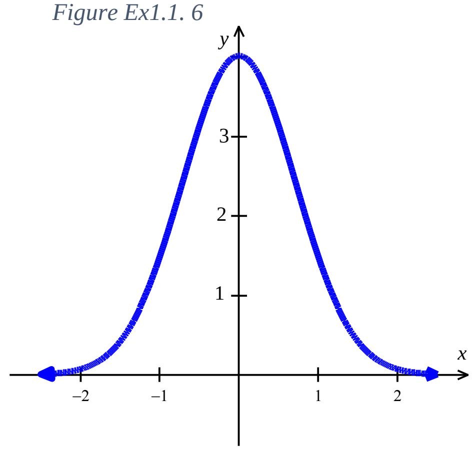
Figure Ex1.1. 6



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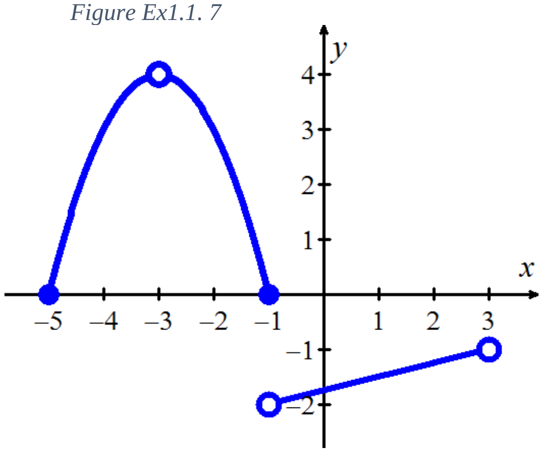

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Figure Ex1.1. 8

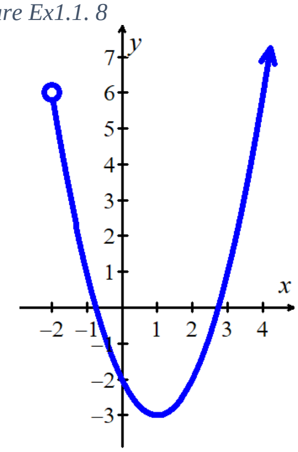

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Figure Ex1.1. 9

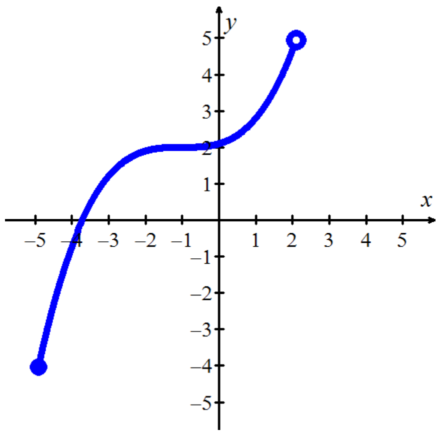

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Figure Ex1.1. 10

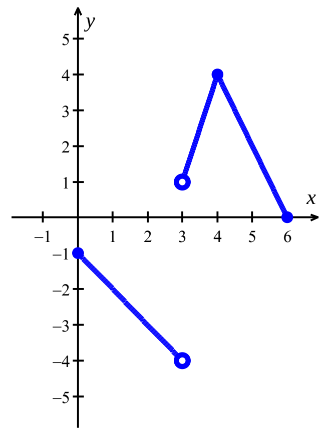

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Figure Ex1.1. 11

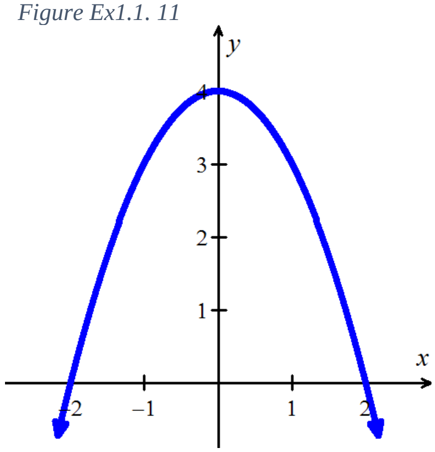

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Figure Ex1.1. 12

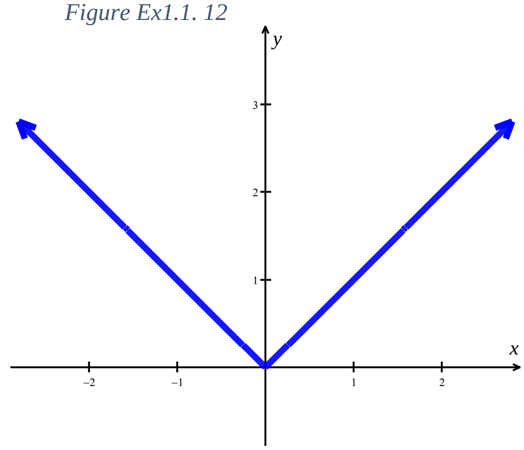

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Figure Ex1.1. 13

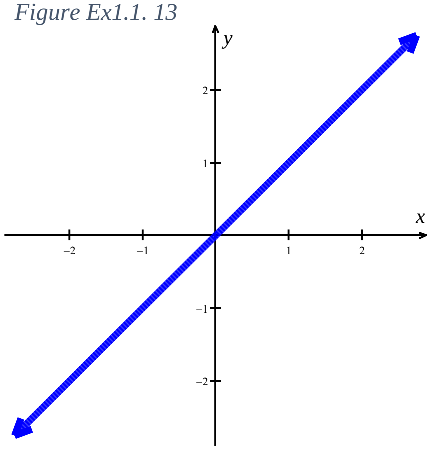

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Figure Ex1.1. 14

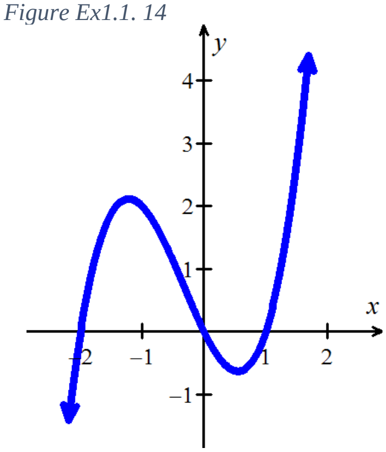

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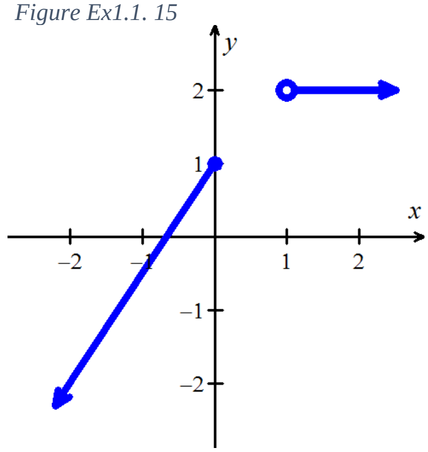
Figure Ex1.1. 15



24.

Figure Ex1.1. 16

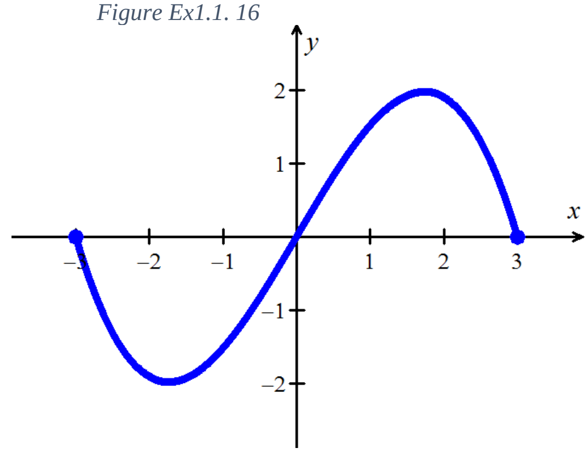


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Figure Ex1.1. 17

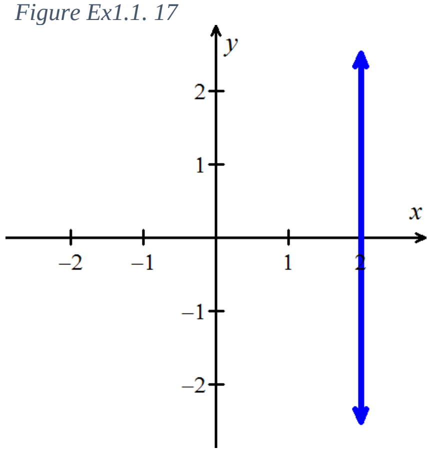

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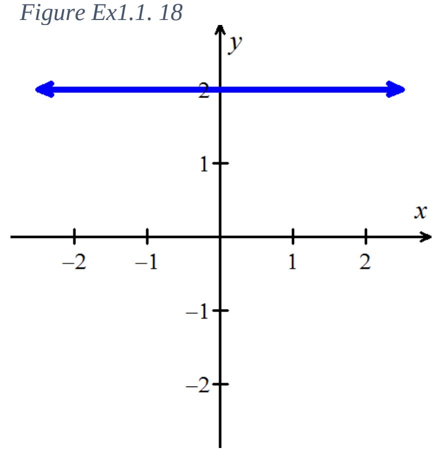
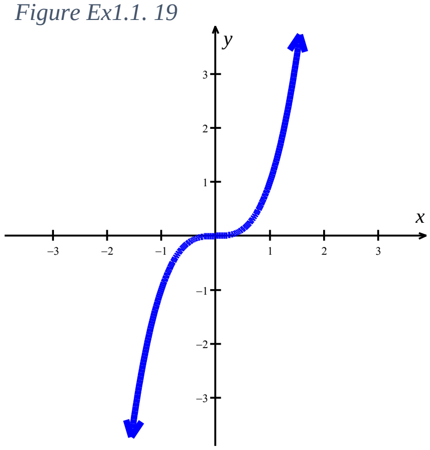
Figure Ex1.1. 18



27.

Figure Ex1.1. 19



28.

Figure Ex1.1. 20

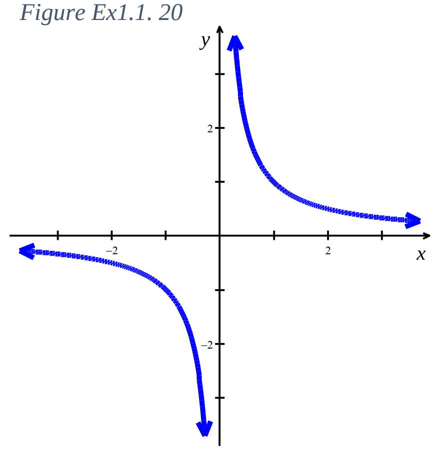


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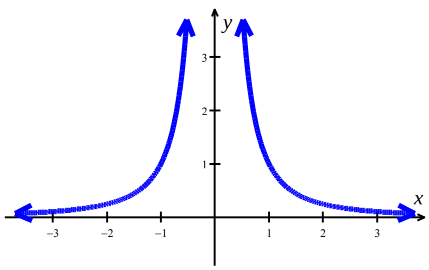
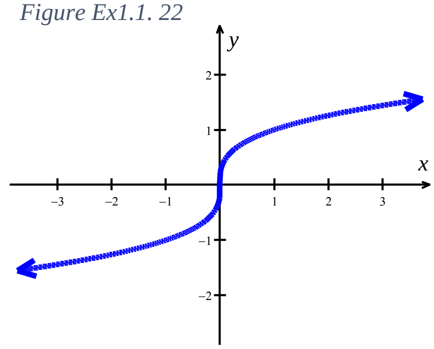
Figure Ex1.1. 21



30.

Figure Ex1.1. 22



In Exercises 31 – 45, determine if the relation, expressed as an equation, is a function. Assume that x is the independent variable and y is the dependent variable. Rewrite the equation as necessary to verify your conclusion that y is, or is not, a function of x.

31. $y = x^3 - x$

32. $y = \sqrt{x-2}$

33. $x^3 y = -4$

34. $x^2 - y^2 = 1$

35. $y = \dfrac{x}{x^2 - 9}$

36. $x = -6$

37. $x = y^2 + 4$

38. $y = x^2 + 4$

39. $x^2 + y^2 = 4$

40. $y = \sqrt{4 - x^2}$

41. $x^2 - y^2 = 4$

42. $x^3 + y^3 = 4$

43. $2x + 3y = 4$

44. $2xy = 4$

45. $x^2 = y^2$

In Exercises 46 – 55, use the given function f to evaluate, if possible, and simplify the following:

(a) $f(3)$ (b) $f(-1)$ (c) $f\left(\dfrac{3}{2}\right)$ (d) $f(a)$

46. $f(x) = 2x + 1$

47. $f(x) = 3 - 4x$

48. $f(x) = 2 - x^2$

49. $f(x) = x^2 - 3x + 2$

50. $f(x) = \dfrac{x}{x-1}$

51. $f(x) = \dfrac{2}{x^3}$

52. $f(x) = \sqrt{x-2}$

53. $f(x) = \sqrt[3]{x}$

54. $f(x) = 6$

55. $f(x) = 0$

In Exercises 56 – 63, use the given function f to evaluate, if possible, and simplify the following:

(a) $f(2)$ (b) $f(-2)$ (c) $f(h)$ (d) $f(0)$

56. $f(x)=2x-5$

57. $f(x)=5-2x$

58. $f(x)=2x^2-1$

59. $f(x)=3x^2+3x-2$

60. $f(x)=\sqrt{2x+1}$

61. $f(x)=117$

62. $f(x)=\dfrac{x}{2}$

63. $f(x)=\dfrac{2}{x}$

In Exercises 64 – 67, use the given function f to solve $f(x)=4$.

64. $f(x)=2x-8$

65. $f(x)=7-3x$

66. $f(x)=2x^2-14$

67. $f(x)=x^2+x+2$

In Exercises 68 – 75, use the given function f to find $f(0)$ and to solve $f(x)=0$, if possible. Then use your answers to write the resulting ordered pairs.[8]

68. $f(x)=2x-1$

69. $f(x)=3-\dfrac{2}{5}x$

70. $f(x)=2x^2-6$

71. $f(x)=x^2-x-12$

72. $f(x)=\sqrt{x+4}$

73. $f(x)=\sqrt{1-2x}$

74. $f(x)=\dfrac{3}{4-x}$

75. $f(x)=\dfrac{3x}{4-x^2}$

In Exercises 76 – 93, find the domain of the function.

76. $f(x)=x^4-13x^3+56x^2-19$

77. $f(x)=x^4+4$

78. $f(x)=\dfrac{x-2}{x+1}$

79. $f(x)=\dfrac{3x}{x^2+x-2}$

[8] Note that you have found points on the *x*- and *y*-axes. These are referred to as *x*- and *y*-intercepts, respectively. We will talk more about this soon.

80. $f(x)=\dfrac{2x}{x^2+4}$

81. $f(x)=\dfrac{2x}{x^2-4}$

82. $f(x)=\dfrac{x+4}{x^2-36}$

83. $f(x)=\dfrac{x-2}{x-2}$

84. $f(x)=\sqrt{3-x}$

85. $f(x)=\sqrt{2x+5}$

86. $f(x)=\sqrt{x+3}$

87. $f(x)=\dfrac{\sqrt{7-x}}{x^2+1}$

88. $f(x)=\sqrt{6x-2}$

89. $f(x)=\dfrac{6}{\sqrt{6x-2}}$

90. $f(x)=\sqrt[3]{6x-2}$

91. $f(x)=\dfrac{\sqrt{6x-2}}{x^2-36}$

92. $s(t)=\dfrac{t}{t-8}$

93. $Q(r)=\dfrac{\sqrt{r}}{r-8}$

1.2 Graphs of Functions

Learning Objectives

- Solve real-world applications of piecewise-defined functions.
- Identify and graph the toolkit/parent functions.
- Graph piecewise-defined functions.
- Determine whether a function is even, odd, or neither.
- Determine where a function is increasing, decreasing, or constant.
- Determine local maxima and local minima.
- Determine absolute maximum and absolute minimum values.

Through mathematics, it is possible to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends. In each of these scenarios, functions play an important role. We begin this section by looking at some real-world applications of functions.

Modeling with Functions

It is important to keep in mind that any time mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for \$1.50 per pound. To develop a formula that relates the cost of buying grapes to the amount of grapes being purchased, we let the variable c denote the cost of the grapes and the variable g denote the amount of grapes purchased. To find the cost of the grapes, we use the formula $c = 1.5g$.

We can think of g as the independent variable and c as the dependent variable, and write our formula in function notation: $c(g) = 1.5g$. With g representing the amount of grapes purchased (in pounds) and $c(g)$ representing the cost (in dollars), we next determine the **applied domain**[9] of this function. Even though, mathematically, $c(g) = 1.5g$ has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of g that do not make any physical sense. For example, $g = -1$ corresponds to purchasing −1 pounds of grapes.[10] Also, unless the local market mentioned is the State of California (or some other exporter of grapes), it doesn't make sense for $g = 500{,}000{,}000$. The

[9] Also known as the **explicit domain**.
[10] Maybe this means returning a pound of grapes?

reality of the situation limits what g can be, and these limits determine the applied domain. Typically, an applied domain is stated explicitly. In this case it would be common to see something like $c(g) = 1.5g$, $0 \le g \le 100$, meaning the number of pounds of grapes purchased is limited from 0 up to 100.

Example 1.2.1. The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t & \text{if } 0 \le t \le 20 \\ 0 & \text{if } t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.
2. Solve $h(t) = 375$ and interpret your answer.

Solution.

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20, inclusive, and another for values of t greater than 20.

   Since $t = 10$ satisfies the inequality $0 \le t \le 20$, we use the first formula listed to find $h(10)$.

$$\begin{aligned} h(t) &= -5t^2 + 100t \\ h(10) &= -5(10)^2 + 100(10) \\ &= 500 \end{aligned}$$

   With t representing the number of seconds since lift-off and $h(t)$ the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground.

   To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the formula $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.

2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces.

   For $0 \le t \le 20$, $h(t) = -5t^2 + 100t$, so for these values of t, we solve $-5t^2 + 100t = 375$.

$$
\begin{aligned}
-5t^2+100t &= 375 \\
-5t^2+100t-375 &= 0 \\
-5\left(t^2-20t+75\right) &= 0 \\
-5(t-5)(t-15) &= 0
\end{aligned}
$$

Setting each factor equal to 0, we get $t=5$ and $t=15$. Both of these values of t lie between 0 and 20, so are solutions.

For $t>20$, $h(t)=0$ and, in this case, there are no solutions to $0=375$.

In terms of the model rocket, $h(t)=375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t=5$ and $t=15$, correspond to the rocket reaching this altitude twice: once 5 seconds after launch (on the way up) and again 15 seconds after launch (on the way down).

□

The type of function in the previous example is called a **piecewise-defined function**, or **piecewise function** for short. Many real-world phenomena (income tax formulas, for example) are modeled by piecewise-defined functions. A piecewise-defined function uses more than one formula to define the output, and each formula has its own domain. The domain of the function is the union of all of these smaller domains. Visualizing piecewise functions through graphing is helpful. We will return to piecewise-defined functions and their graphs following the introduction of some basic functions.

The Toolkit Functions

We next introduce the basic functions that are referred to as **toolkit functions**, or **parent functions**. These functions, combinations of these functions, their graphs, and transformations, will be used frequently throughout our study of College Algebra. It is helpful to recognize these functions quickly by name, formula, and graph, and to know their properties such as domain, range, and intercepts. Note the following definition for intercepts.

Definition 1.4. Suppose the graph of an equation is given.

- A point on the graph that is also on the *x*-axis is called an ***x*-intercept** of the graph.
- A point on the graph that is also on the *y*-axis is called a ***y*-intercept** of the graph.

Given an equation rather than a graph, since *x*-intercepts occur when $y=0$ and *y*-intercepts occur when $x=0$, we can find *x*-intercepts by setting y equal to zero, and *y*-intercepts by setting x equal to zero.

Example 1.2.2. Find the x- and y-intercepts of the graph of the function $f(x) = -3x + 4$.

Solution. To determine the x-intercept(s), we set y equal to zero. In this case $y = f(x)$, so we set $f(x)$ equal to zero, where $f(x) = -3x + 4$.

$$\begin{aligned} -3x + 4 &= 0 \\ -3x &= -4 \\ x &= \frac{4}{3} \end{aligned}$$

The x-intercept is the point $\left(\frac{4}{3}, 0\right)$. To find the y-intercept, we set x equal to zero and have $y = f(0) = -3(0) + 4$, resulting in $y = 4$. Thus, the y-intercept is $(0, 4)$.

□

The toolkit (parent) functions follow, along with a few sample values, graphs, and properties.

1. The Constant Function: $f(x) = c$, where c is a constant

x	$f(x) = c$
-2	c
-1	c
0	c
1	c
2	c

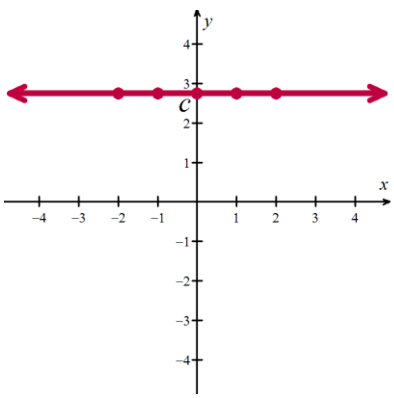


Figure 1.2. 1

Constant Function
$f(x) = c$, where c is a constant

Domain: $(-\infty, \infty)$
Range: $[c, c]$, or $\{c\}$
x-intercepts: none if $c \neq 0$;
$\{(x, 0) \mid x \text{ is a real number}\}$ if $c = 0$
y-intercept: $(0, c)$

The domain of the **constant function**, $f(x) = c$, consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. To find the x-intercepts, we set $f(x) = 0$. This has a solution only in the special case where $c = 0$, and in this case x is any real number; the x-intercepts are $(x, 0)$. To find the y-intercept, we set $x = 0$ to get $y = f(0) = c$, for the y-intercept of $(0, c)$.

2. The Identity Function: $f(x) = x$

x	$f(x) = x$
-2	-2
-1	-1
0	0
1	1
2	2

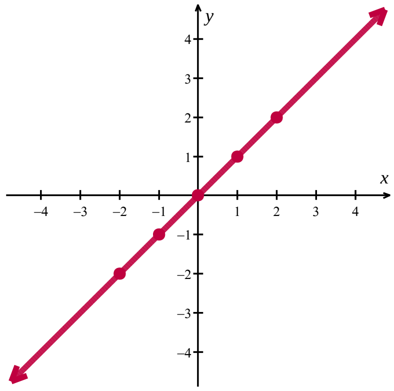


Figure 1.2. 2

Identity Function
$f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

x- and *y*-intercept: $(0,0)$

For the **identity function**, $f(x) = x$, there is no restriction on x, and for this function the output is the same as the input. Thus, both the domain and range are the set of all real numbers. To find the *x*-intercepts, we set $f(x) = 0$, which happens only when $x = 0$. For the *y*-intercept we set $x = 0$ to get $y = f(0) = 0$. So, the point $(0,0)$ is both the *x*- and *y*-intercept.

3. The Absolute Value Function: $f(x) = |x|$

x	$f(x) = \|x\|$
-2	2
-1	1
0	0
1	1
2	2

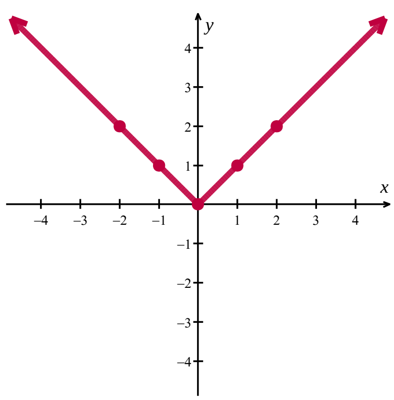


Figure 1.2. 3

Absolute Value Function
$f(x) = |x|$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

x- and *y*-intercept: $(0,0)$

There is no restriction on x for the **absolute value function**, $f(x) = |x|$. However, the output can only be greater than or equal to zero. For the *x*-intercept, $|x| = 0$ occurs only when $x = 0$. The *y*-intercept is at $y = |0| = 0$. Thus, the point $(0,0)$ is both the *x*- and *y*-intercept.

We note that $f(x) = |x|$ can be defined as a piecewise-defined function: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

4. The Squaring Function: $f(x)=x^2$

x	$f(x)=x^2$
-2	4
-1	1
0	0
1	1
2	4

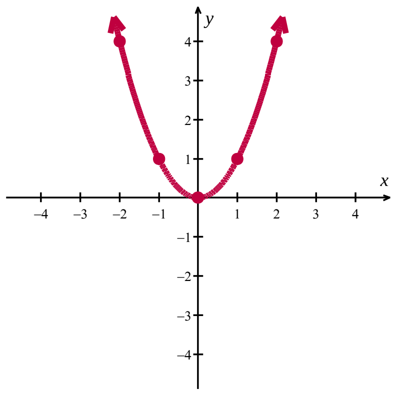


Figure 1.2. 4

Squaring Function
$f(x)=x^2$

Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

x- and *y*-intercept: $(0,0)$

For the **squaring function**, $f(x)=x^2$, input values include all real numbers. Because output values are positive or zero, the range includes only nonnegative real numbers and, as seen from the graph, is all nonnegative real numbers. For the *x*-intercept, we set $x^2=0$ and find $x=\pm\sqrt{0}=0$. To find the *y*-intercept, we set $x=0$ to get $y=0^2=0$. The point $(0,0)$ is both the *x*- and *y*-intercept.

5. The Cubing Function: $f(x)=x^3$

x	$f(x)=x^3$
-2	-8
-1	-1
0	0
1	1
2	8

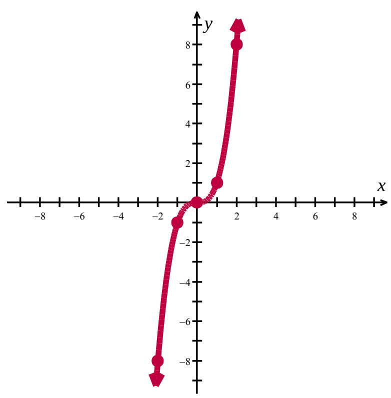


Figure 1.2. 5

Cubing Function
$f(x)=x^3$

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

x- and *y*-intercept: $(0,0)$

For the **cubing function**, $f(x)=x^3$, input values include all real numbers and, as seen from the graph, the range is also all real numbers. For the *x*-intercept, we set $x^3=0$ and find $x=\sqrt[3]{0}=0$. For the *y*-intercept, we set $x=0$ to find $y=0^3=0$. The point $(0,0)$ is both the *x*- and *y*-intercept.

6. The Reciprocal Function: $f(x)=\frac{1}{x}$

x	$f(x)=\frac{1}{x}$
-2	$-\frac{1}{2}$
-1	-1
0	undefined
1	1
2	$\frac{1}{2}$

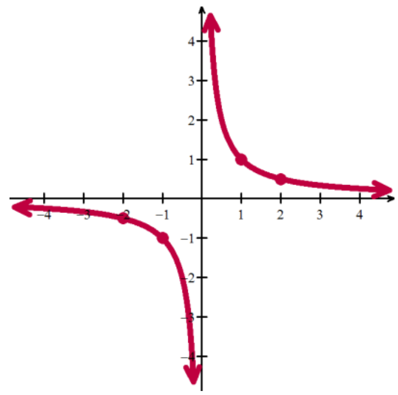


Figure 1.2. 6

Reciprocal Function

$f(x)=\frac{1}{x}$

Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(-\infty,0)\cup(0,\infty)$

Intercepts: none

For the **reciprocal function**, $f(x)=\frac{1}{x}$, we cannot divide by zero so we must exclude zero from the domain. Further, $\frac{1}{x}=0$ has no solution so zero is not in the range. There are no *x*-intercepts, given that $\frac{1}{x}=0$ has no solution. There is no *y*-intercept since $y=\frac{1}{0}$ is undefined.

7. The Reciprocal Squared Function: $f(x)=\frac{1}{x^2}$

x	$f(x)=\frac{1}{x^2}$
-2	$\frac{1}{4}$
-1	1
0	undefined
1	1
2	$\frac{1}{4}$

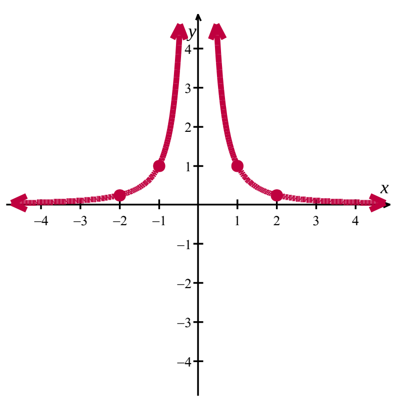


Figure 1.2. 7

Reciprocal Squared Function

$f(x)=\frac{1}{x^2}$

Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(0,\infty)$

Intercepts: none

For the **reciprocal squared function**, $f(x)=\frac{1}{x^2}$, with reasoning similar to that for the reciprocal function, we find the domain and range exclude zero. The output of this function is always positive due to the square in the denominator and, as seen from the graph, the range is the set of all positive real numbers. There are no *x*-intercepts since $\frac{1}{x^2}=0$ has no solution, and no *y*-intercept since $y=\frac{1}{0^2}$ is undefined.

8. The Square Root Function: $f(x)=\sqrt{x}$

x	$f(x)=\sqrt{x}$
-4	undefined
-1	undefined
0	0
1	1
4	2

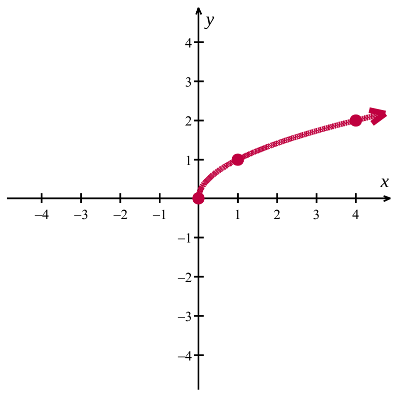


Figure 1.2. 8

Square Root Function

$f(x)=\sqrt{x}$

Domain: $[0,\infty)$

Range: $[0,\infty)$

x- and *y*-intercept: $(0,0)$

For the **square root function**, $f(x)=\sqrt{x}$, since the square root of a negative real number is not a real number, the domain includes only nonnegative numbers. The range also excludes negative numbers because the square root of a positive number x is defined to be positive and the square root of zero is zero, all of which are nonnegative. As seen from the graph, the range is all nonnegative real numbers. The *x*-intercept occurs when $\sqrt{x}=0$, which happens when $x=0$. We find the *y*-intercept by setting x equal to zero: $y=\sqrt{0}=0$. The point $(0,0)$ is both the *x*- and *y*-intercept.

9. The Cube Root Function: $f(x)=\sqrt[3]{x}$

x	$f(x)=\sqrt[3]{x}$
-8	-2
-1	-1
0	0
1	1
8	2

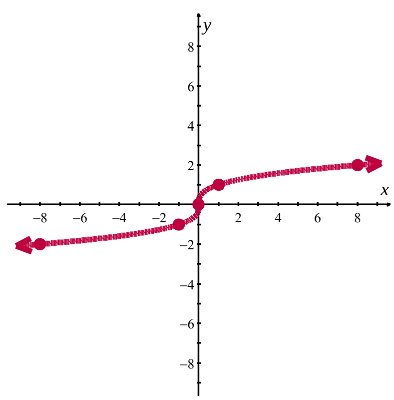


Figure 1.2. 9

Cube Root Function

$f(x)=\sqrt[3]{x}$

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

x- and *y*-intercept: $(0,0)$

We note that there is no problem taking the cube root of a negative number, and that the resulting output is negative. So, for the **cube root function**, $f(x)=\sqrt[3]{x}$, input values include all real numbers and, as seen from the graph, the range is also all real numbers. We find that the *x*-intercept occurs when $\sqrt[3]{x}=0$, so that $x=0^3=0$. For the *y*-intercept, we set x equal to zero to find $y=\sqrt[3]{0}=0$. The point $(0,0)$ is both the *x*- and *y*-intercept.

In finding the *x*-intercepts of a function f, we note that the *x*-coordinates are found by solving $f(x)=0$. The solutions to $f(x)=0$ are called zeros of the function. If a solution x has a real value, then $(x,0)$ is an *x*-intercept of the graph of f. We state the definition of zeros before moving on.

Definition 1.5. The **zeros** of a function f are the solutions to the equation $f(x)=0$.

Graphing Piecewise-Defined Functions

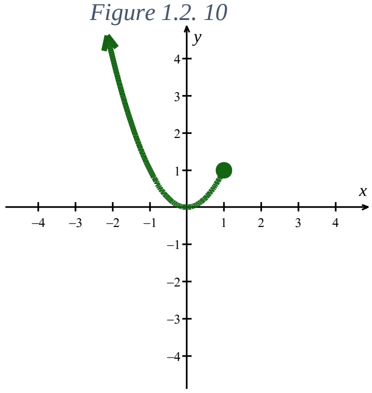
Graphing piecewise-defined functions is a bit of a challenge, but familiarity with the toolkit functions will be helpful.

Example 1.2.3. Sketch a graph of the function.

$$f(x)=\begin{cases} x^2 & \text{if} \quad x\le 1 \\ 3 & \text{if} \quad 1<x\le 2 \\ x & \text{if} \quad x>2 \end{cases}$$

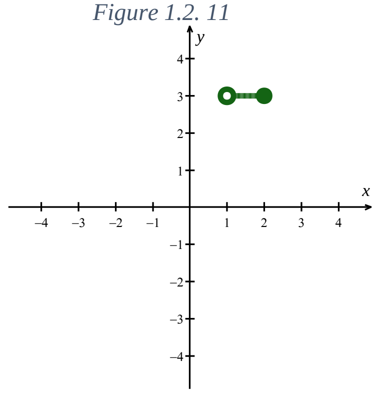
Solution. Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw an open circle to indicate where the endpoint is not included because of a 'less than' or 'greater than' inequality; we draw a filled-in circle where the endpoint is included because of a 'less than or equal to' or 'greater than or equal to' inequality. Following are the three components of this piecewise-defined function, each graphed separately.

Figure 1.2. 10



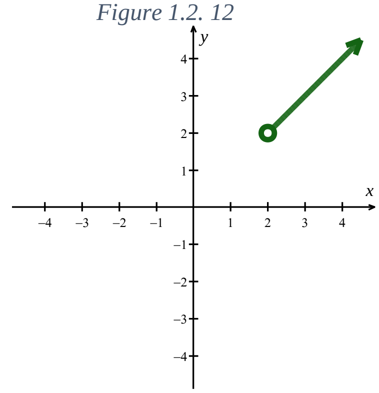
$f(x)=x^2$ if $x\le 1$

Figure 1.2. 11



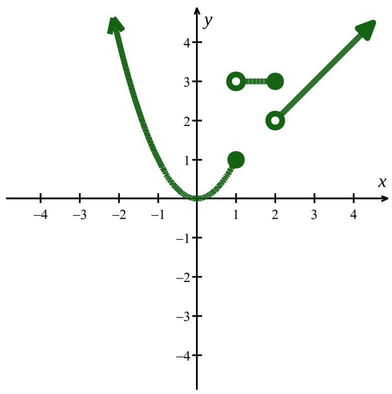
$f(x)=3$ if $1<x\le 2$

Figure 1.2. 12



$f(x)=x$ if $x>2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane.



$$f(x)=\begin{cases} x^2 & \text{if} \quad x\le 1 \\ 3 & \text{if} \quad 1<x\le 2 \\ x & \text{if} \quad x>2 \end{cases}$$

Figure 1.2. 13

□

Symmetry

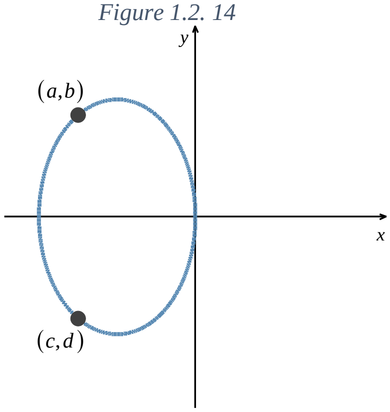
In the Cartesian plane, a graph is said to be symmetric about the *x*-axis, *y*-axis, or origin if, for each point (a,b) on the graph, there is a point (c,d) on the graph that is symmetric to point (a,b) about the *x*-axis, *y*-axis, or origin, respectively, according to the following definition.

Definition 1.6. Two points (a,b) and (c,d) in the plane are said to be

- symmetric **about the *x*-axis** if $a=c$ and $b=-d$;
- symmetric **about the *y*-axis** if $a=-c$ and $b=d$;
- symmetric **about the origin** if $a=-c$ and $b=-d$.

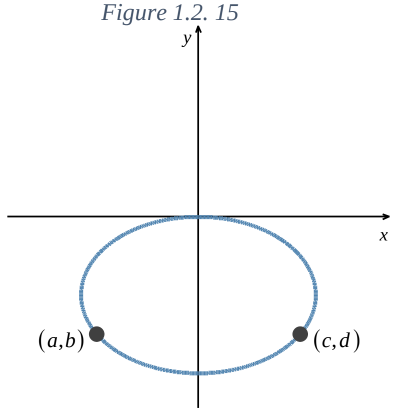
These three types of symmetry are demonstrated below, with sample points (a,b) and (c,d) .

Figure 1.2. 14



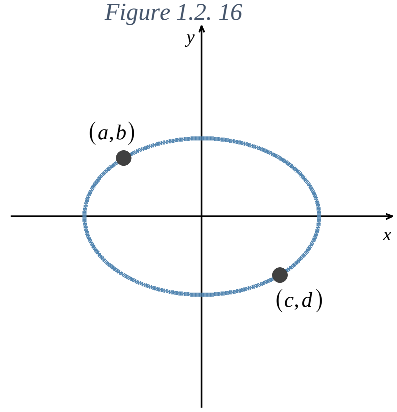
Symmetry about *x*-axis
($a=c$ and $b=-d$)

Figure 1.2. 15



Symmetry about *y*-axis
($a=-c$ and $b=d$)

Figure 1.2. 16



Symmetry about origin
($a=-c$ and $b=-d$)

Note that these three graphs do not represent functions. We next look at the association of symmetry in the graph of a function with the property of the function being ‘even’ or ‘odd’.

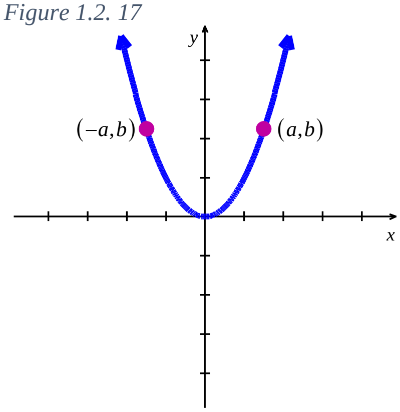
Even and Odd Functions

We refer to functions as being 'even' or 'odd' if they possess symmetry about the *y*-axis or origin, respectively.

> **Definition 1.7.** A function f is called
> - **even** if its graph is symmetric about the *y*-axis;
> - **odd** if its graph is symmetric about the origin.

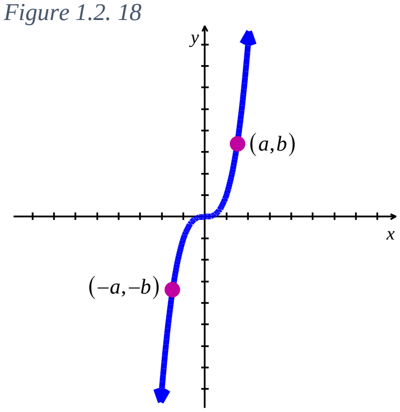
The graphs of the toolkit functions $f(x)=x^2$ and $f(x)=x^3$ follow.

Figure 1.2. 17



$f(x)=x^2$

Figure 1.2. 18



$f(x)=x^3$

Note that for each point (a,b), there is a corresponding point $(-a,b)$ on the graph of $f(x)=x^2$, so the graph is symmetric about the *y*-axis and the function is even. For $f(x)=x^3$, the graph contains a point $(-a,-b)$ for each point (a,b); thus the graph is symmetric about the origin and the function is odd.

To analytically test if the graph of a function, $y=f(x)$, is symmetric about the *y*-axis, we replace x with $-x$, resulting in the equation $y=f(-x)$. For the graph of $y=f(x)$ to be symmetric about the *y*-axis, we must have $f(-x)=f(x)$. In a similar fashion, to test the function $y=f(x)$ for symmetry about the origin, we replace x with $-x$ and y with $-y$. Doing this substitution in the equation $y=f(x)$ results in $-y=f(-x)$. Then, solving for *y* gives $y=-f(-x)$. For the graph of $y=f(x)$ to be symmetric about the origin, we must have $-f(-x)=f(x)$ or, equivalently, $f(-x)=-f(x)$. Following is a summary of these results.

Testing a Function for Symmetry

The graph of a function f is symmetric

- about the *y*-axis if and only if $f(-x) = f(x)$ for all x in the domain of f.
- about the origin if and only if $f(-x) = -f(x)$ for all x in the domain of f.

Apart from a very specialized family of functions that are both even and odd, functions fall into one of three distinct categories: even, odd, or neither even nor odd.

Example 1.2.4. Determine analytically if the following functions are even, odd, or neither even nor odd.

1. $f(x) = x^3 + 2x$
2. $g(x) = \dfrac{5}{2 - x^2}$
3. $h(x) = x^2 - \dfrac{x}{100} - 1$

Solution.

1. The first step in determining whether $f(x) = x^3 + 2x$ is even or odd is to replace x with $-x$ and simplify.

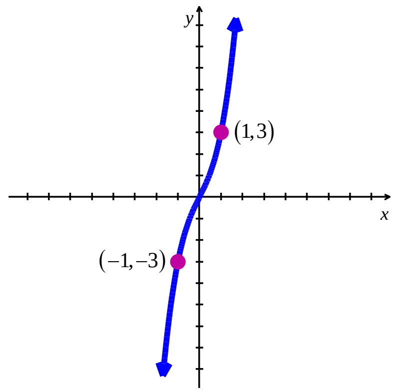
$$\begin{aligned} f(-x) &= (-x)^3 + 2(-x) \\ &= -x^3 - 2x \end{aligned}$$

It does not appear that $f(x) = f(-x)$. To prove this, we can check with an *x*-value. After some trial and error, we see that $f(1) = 3$ and $f(-1) = -3$. This proves that f is not even, but it does not rule out the possibility that f is odd. To check if f is odd, we compare $f(-x)$ with $-f(x)$.

$$\begin{aligned} -f(x) &= -\left(x^3 + 2x\right) \\ &= -x^3 - 2x \end{aligned}$$

Because $f(-x) = -f(x)$, f is an odd function. Notice that the following graph of f appears to be symmetric about the origin. However, be cautious! As we will see in part 3 of this example, the appearance of symmetry cannot always be trusted. In this case, since we have applied the test for symmetry to verify that f is an odd function, we can be certain that its graph is symmetric about the origin. Just keep in mind that we must rely on the test for symmetry to verify that symmetry exists.

Figure 1.2. 19



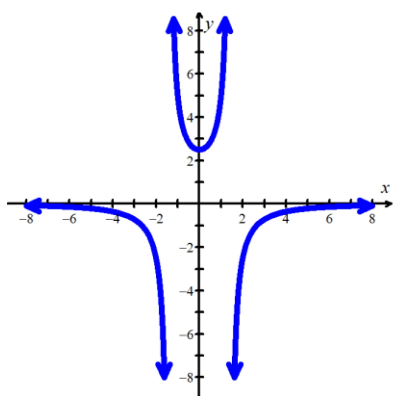
$$y = f(x) = x^3 + 2x$$

2. To determine if $g(x) = \dfrac{5}{2-x^2}$ is even or odd, we begin by finding $g(-x)$.

$$g(-x) = \frac{5}{2-(-x)^2}$$
$$= \frac{5}{2-x^2}$$

Since $g(-x) = g(x)$, g is even. The following graph of $g(x)$ matches this result in being symmetric about the *y*-axis.

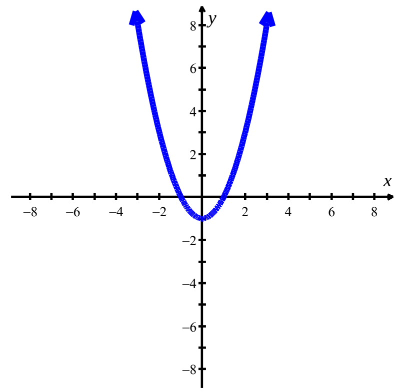
Figure 1.2. 20



$$y = g(x) = \frac{5}{2-x^2}$$

3. For the third function, to demonstrate the need for using a test to verify symmetry, we begin by showing the graph of $h(x) = x^2 - \dfrac{x}{100} - 1$.

Figure 1.2. 21



$$y = h(x) = x^2 - \frac{x}{100} - 1$$

While the graph appears to represent an even function, we can draw no conclusions without verifying that $h(x) = h(-x)$. We can, however, show that h is **not** even by providing a single value of x for which $h(x) \neq h(-x)$. If we set $x = 1$, we get $h(1) = -\frac{1}{100}$ and $h(-1) = \frac{1}{100}$. Thus, $h(1) \neq h(-1)$ and this one value of x confirms that h is not even.

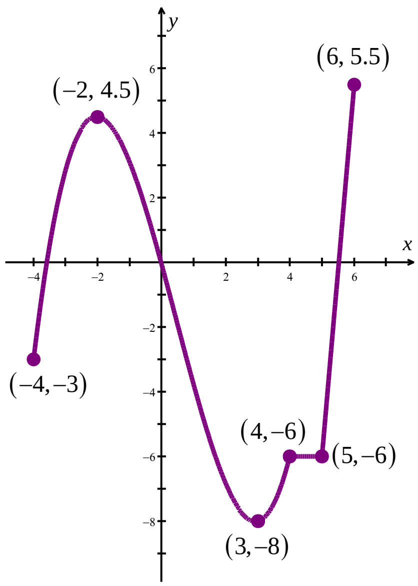
From the graph, it does not appear that h is odd. We verify that this is the case by searching for a single value of x for which $h(-x) \neq -h(x)$. For the value $x = 2$, we find $h(-2) = \frac{151}{50}$ and $-h(2) = -\frac{149}{50}$, so $h(-2) \neq -h(2)$. This verifies that h is not odd. We conclude that h is neither even nor odd.

□

Determining Where a Function is Increasing, Decreasing or Constant

Consider the graph of the following function, $y = f(x)$.

Figure 1.2. 22



Reading from left to right, the graph $y = f(x)$ starts at the point $(-4, -3)$ and ends at the point $(6, 5.5)$. If we imagine walking from left to right on the graph, then

- between $(-4, -3)$ and $(-2, 4.5)$, we are walking uphill;
- between $(-2, 4.5)$ and $(3, -8)$, we are walking downhill;
- between $(3, -8)$ and $(4, -6)$, we are walking uphill once more;
- from $(4, -6)$ to $(5, -6)$, we level off;
- from $(5, -6)$ to $(6, 5.5)$, we resume walking uphill.

In other words, for the *x*-values between -4 and -2, the *y*-coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since $y = f(x)$, the *y*-values on the graph are the function values, and we say that the function f is **increasing** on the interval $(-4, -2)$. Analogously, we say that f is **decreasing** on the interval $(-2, 3)$, increasing once more on the interval $(3, 4)$, **constant** on the interval $(4, 5)$, and finally increasing once again on the interval $(5, 6)$.

It is extremely important to notice that the behavior (increasing, decreasing, or constant) occurs on an interval on the *x*-axis. When we say that the function f is increasing on $(-4, -2)$, we mean for *x*-values between −4 and −2 and we do not mention the actual *y*-values along the way. Thus, we report where the behavior occurs, not to what extent the behavior occurs.[11] We are now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing, or constant.

Definition 1.8. Suppose f is a function defined on an open interval I. We say f is

- **increasing** on I if and only if, for all real numbers a and b in I with $a < b$, $f(a) < f(b)$.
- **decreasing** on I if and only if, for all real numbers a and b in I with $a < b$, $f(a) > f(b)$.
- **constant** on I if and only if, for all real numbers a and b in I, $f(a) = f(b)$.

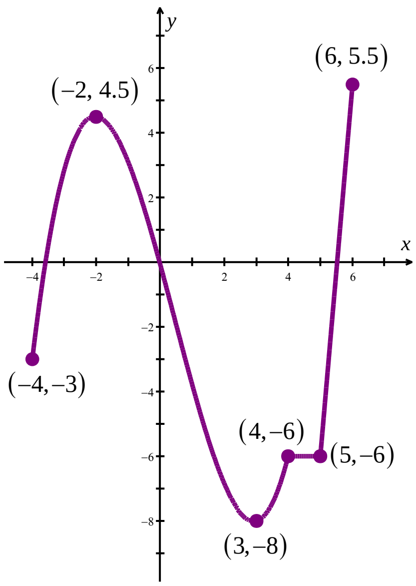
[11] The notion of how quickly or how slowly a function increases or decreases is explored in Calculus.

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing, and constant agree with our graphical descriptions earlier.

Maximum and Minimum Function Values

Now let us turn our attention to a few of the points on the graph of $y = f(x)$.

Figure 1.2. 23



Clearly the point $(-2, 4.5)$ does not have the largest *y*-value. Indeed, that honor goes to $(6, 5.5)$. But $(-2, 4.5)$ should get some sort of consolation prize for being at the 'top of the hill' between $x = -4$ and $x = 3$. We say that the function f has a **local maximum**[12] at the point $(-2, 4.5)$ because 4.5 is the largest *y*-value (hence, function value) on the curve near $x = -2$. We say that this **local maximum value** is 4.5. Similarly, the function f has a **local minimum**[13] at the point $(3, -8)$ since the *y*-coordinate -8 is the smallest function value near $x = 3$. That **local minimum value** is -8. As we will see in the next definition, we will not classify the endpoints $(-4, -3)$ and $(6, 5.5)$ as local minimum or local maximum points.

Some important terminology to become familiar with is **maxima**, which is the plural of maximum, and **minima**, the plural of minimum. **Extrema** is the plural of **extremum**, which includes both maximum and minimum. We have one last observation to make before we proceed to the algebraic definitions and look at an example.

If we look at the entire graph of f, we see that the largest *y*-value (the largest function value) is 5.5 at $x = 6$. In this case, we say the **absolute maximum value**[14] of f is 5.5. Similarly, the **absolute minimum value**[15] of f is -8.

We formalize these concepts in the following definitions.

[12] Also called a **relative maximum**.
[13] Also called a **relative minimum**.
[14] Sometimes called **global maximum**.
[15] Sometimes called **global minimum**.

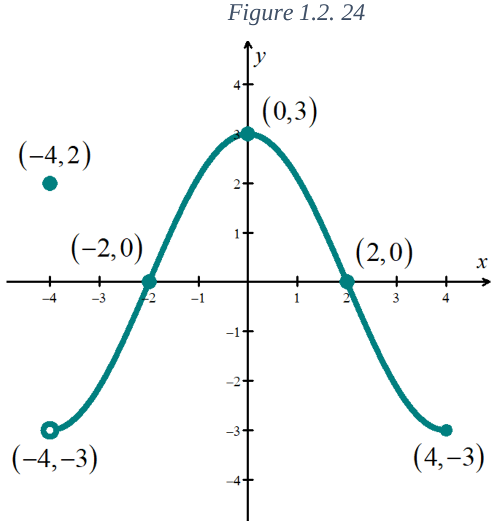
Definition 1.9. Suppose f is a function with $f(a)=b$.

- We say the point (a,b) is a **local maximum** point of f if there is an open interval I containing a for which $f(a)\geq f(x)$ for all x in I. The value $f(a)=b$ is called a local maximum value of f.
- We say the point (a,b) is a **local minimum** point of f if there is an open interval I containing a for which $f(a)\leq f(x)$ for all x in I. The value $f(a)=b$ is called a local minimum value of f.
- The value $f(a)=b$ is called the **absolute maximum value** of f if $b\geq f(x)$ for all x in the domain of f.
- The value $f(a)=b$ is called the **absolute minimum value** of f if $b\leq f(x)$ for all x in the domain of f.

As mentioned earlier, the above definition does not allow for a local maximum, or local minimum, to occur at the end point of an interval since we require $f(a)$ to be the largest, or smallest, value of the function on an open interval containing a. Otherwise stated, $f(a)$ must be the largest, or smallest, value of the function for *x*-values on both sides of a. However, an endpoint may be the point where an absolute maximum or absolute minimum value occurs. This concept is addressed in the following example, along with a summary of other concepts.

Example 1.2.5. Given the graph of $y=f(x)$ below, answer all of the following questions.

Figure 1.2. 24



1. Find the domain of f.
2. Find the range of f.
3. List the *x*-intercepts, if any exist.
4. List the *y*-intercept, if any exists.
5. Find the real zeros of f.
6. Solve $f(x) < 0$.
7. Determine $f(2)$.
8. Solve $f(x) = -3$.
9. Find the number of solutions to $f(x) = 1$.
10. Does f appear to be even, odd, or neither?
11. List the intervals on which f is increasing.
12. List the intervals on which f is decreasing.
13. List the local maxima, if any exist.
14. List the local minima, if any exist.
15. Find the absolute maximum, if it exists.
16. Find the absolute minimum, if it exists.

Solution.

1. To find the domain of f, we proceed as in **Section 1.1**. By projecting the graph to the *x*-axis, we see that the portion of the *x*-axis that corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the *y*-axis. We see that the *y*-values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. The *x*-intercepts are the points on the graph with *y*-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

4. The *y*-intercept is the point on the graph with *x*-coordinate 0, namely $(0, 3)$.

5. The real zeros of f are the *x*-coordinates of the *x*-intercepts on the graph of $y = f(x)$; that is, $x = -2$ and $x = 2$.

6. To solve $f(x) < 0$, we look for the *x*-values of the points on the graph where the *y*-coordinate is less than 0. Graphically, we are looking for where the graph is below the *x*-axis. This happens for *x*-values from -4 to -2, excluding -4 due to the hole in the graph at that point, and excluding -2 since that point is on the *x*-axis, not below it. Additionally, the graph is below the *x*-axis from 2 to 4, excluding 2. Our answer is $(-4, -2) \cup (2, 4]$.

7. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the *y*-coordinate of the point that corresponds to $x = 2$. Noting that the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

8. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find one point with a *y*-coordinate of -3, namely $(4,-3)$. Hence, the solution to $f(x) = -3$ is $x = 4$.

9. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the *y*-coordinate is 1. Even though these points are not specified, we see that the curve has two points with a *y*-value of 1, as shown in the graph below. That means there are two solutions to $f(x) = 1$.

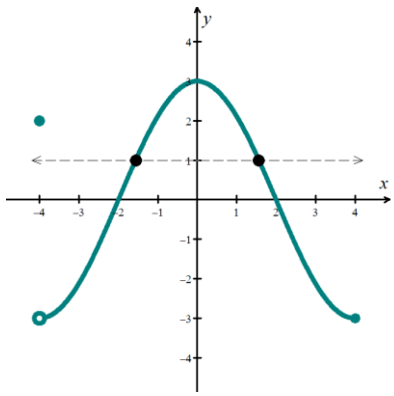


Figure 1.2. 25

10. Were it not for the hole in the graph at the point $(-4,-3)$, and the point $(-4,2)$, the graph would appear to be symmetric about the *y*-axis, in which case the function would be even. However, this graph has no symmetry and the function is neither even nor odd.

11. As we move from left to right, the graph rises from the hole at the point $(-4,-3)$ to the point $(0,3)$. This means f is increasing on the interval $(-4,0)$. (Remember, the answer here is an open interval on the *x*-axis.)

12. As we move from left to right, the graph falls from the point $(0,3)$ to the point $(4,-3)$. This means f is decreasing on the interval $(0,4)$. (Remember, the answer here is an open interval on the *x*-axis.)

13. The function has its only local maximum at the point $(0,3)$, so $f(0) = 3$ is a local maximum value.

14. There are no local minima. Take a closer look at the point $(4,-3)$, which is a low point on the graph. Recall that, in the definition of local minimum, there needs to be an open interval I that contains $x = 4$ such that $f(4) \le f(x)$ for all x in I. But if we put an open interval around

$x = 4$, a portion of that interval will be outside of the domain of f. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(4, -3)$.

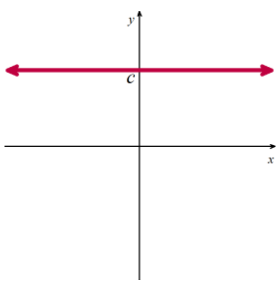
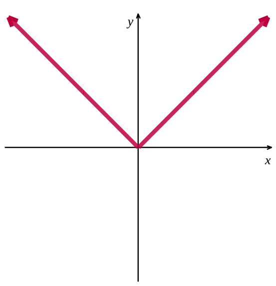
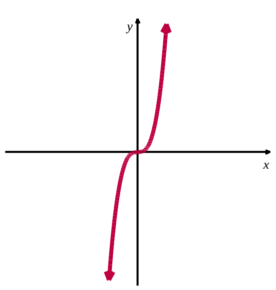
15. The absolute maximum value of f is the largest y-coordinate, which is 3.

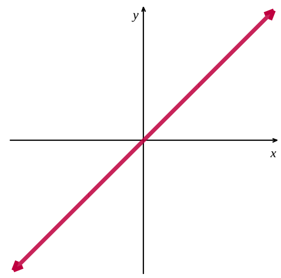
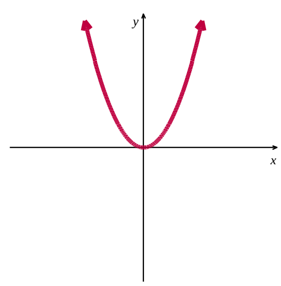
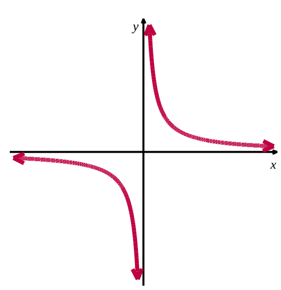
16. The absolute minimum value of f is the smallest y-coordinate, which is -3.

□

Analyzing the Graphical Behavior of the Toolkit Functions

We end this section by revisiting the toolkit functions. We determine whether these functions are even or odd, where their graphs are increasing or decreasing, and values/locations of local and absolute extrema.

 *Figure 1.2. 26*	Constant Function $f(x) = c$ Even (Odd also if $c = 0$) Neither increasing nor decreasing By definition, local and absolute extrema occur at all x-values. The maximum/minimum value is c.
 *Figure 1.2. 28*	Absolute Value Function $f(x) = \lvert x \rvert$ Even Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ A local, and the absolute, minimum value of 0 occurs at $x = 0$.
 *Figure 1.2. 30*	Cubing Function $f(x) = x^3$ Odd Increasing on $(-\infty, \infty)$ There are no local or absolute extrema.

 *Figure 1.2. 27*	Identity Function $f(x) = x$ Odd Increasing on $(-\infty, \infty)$ There are no local or absolute extrema.
 *Figure 1.2. 29*	Squaring Function $f(x) = x^2$ Even Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ A local, and the absolute, minimum value of 0 occurs at $x = 0$.
 *Figure 1.2. 31*	Reciprocal Function $f(x) = \frac{1}{x}$ Odd Decreasing on $(-\infty, 0)$ and on $(0, \infty)$ There are no local or absolute extrema.

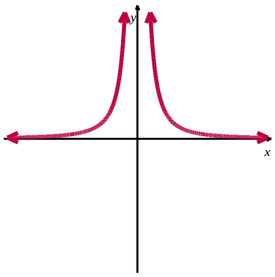


Figure 1.2. 32

Reciprocal Squared Function

$$f(x) = \frac{1}{x^2}$$

Even

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

There are no local or absolute extrema.

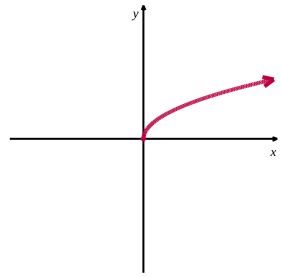


Figure 1.2. 33

Square Root Function

$$f(x) = \sqrt{x}$$

Neither Even nor Odd

Increasing on $(0, \infty)$

There are no local extrema and no absolute maximum. The absolute minimum value of 0 occurs at $x = 0$.

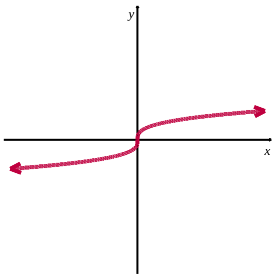


Figure 1.2. 34

Cube Root Function

$$f(x) = \sqrt[3]{x}$$

Odd

Increasing on $(-\infty, \infty)$

There are no local or absolute extrema.

1.2 Exercises

1. How can you determine whether a function is odd or even from the formula for the function?

2. How are the absolute maximum and minimum similar to and different from the local extrema?

3. Compute the following function values for $f(x)=\begin{cases} x+5 & \text{if} \quad x\le -3 \\ \sqrt{9-x^2} & \text{if} \quad -3<x\le 3 \\ -x+5 & \text{if} \quad x>3 \end{cases}$

(a) $f(-4)$ (b) $f(-3)$ (c) $f(3)$

(d) $f(3.001)$ (e) $f(-3.001)$ (f) $f(2)$

4. Compute the following function values for $f(x)=\begin{cases} x^2 & \text{if} \quad x\le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1<x\le 1 \\ x & \text{if} \quad x>1 \end{cases}$

(a) $f(4)$ (b) $f(-3)$ (c) $f(1)$

(d) $f(0)$ (e) $f(-1)$ (f) $f(-0.999)$

In Exercises 5 – 16, sketch the graph of the given piecewise-defined function.

5. $f(x)=\begin{cases} x+1 & \text{if } x<-2 \\ -2x-3 & \text{if } x\ge -2 \end{cases}$

6. $f(x)=\begin{cases} 2x-1 & \text{if } x<1 \\ 1+x & \text{if } x\ge 1 \end{cases}$

7. $f(x)=\begin{cases} x+1 & \text{if } x<0 \\ x-1 & \text{if } x>0 \end{cases}$

8. $f(x)=\begin{cases} 3 & \text{if } x<0 \\ \sqrt{x} & \text{if } x\ge 0 \end{cases}$

9. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ 1-x & \text{if } x>0 \end{cases}$

10. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ x+2 & \text{if } x\ge 0 \end{cases}$

11. $f(x)=\begin{cases} x+1 & \text{if } x<1 \\ x^3 & \text{if } x\ge 1 \end{cases}$

12. $f(x)=\begin{cases} |x| & \text{if } x<2 \\ 1 & \text{if } x\ge 2 \end{cases}$

13. $f(x)=\begin{cases} x^2 & \text{if } x\le 0 \\ 2x & \text{if } x>0 \end{cases}$

14. $f(x)=\begin{cases} -3 & \text{if} \quad x<0 \\ 2x-3 & \text{if} \quad 0\le x\le 3 \\ 3 & \text{if} \quad x>3 \end{cases}$

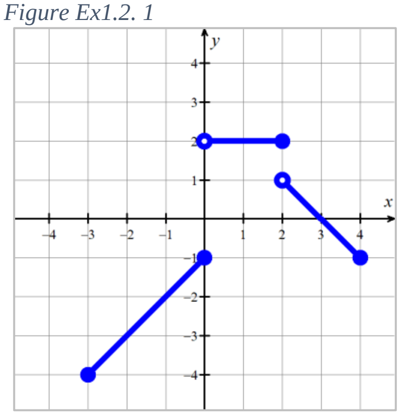
15. $f(x)=\begin{cases} x^2 & \text{if} \quad x \le -2 \\ 3-x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \ge 2 \end{cases}$

16. $f(x)=\begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

In Exercises 17 – 22, complete the description of the piecewise-defined function.

17.

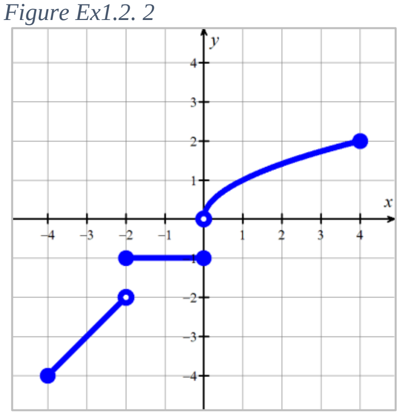
Figure Ex1.2. 1



$$f(x)=\begin{cases} x-1 & \text{if} \quad \underline{\qquad\qquad\qquad} \\ 2 & \text{if} \quad \underline{\qquad\qquad\qquad} \\ 3-x & \text{if} \quad \underline{\qquad\qquad\qquad} \end{cases}$$

18.

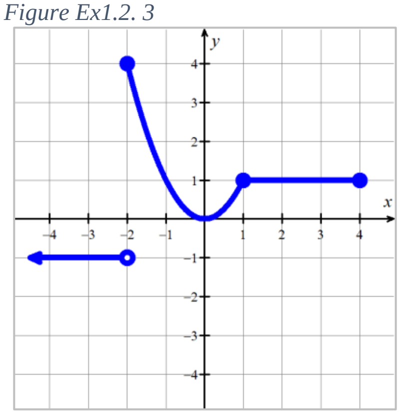
Figure Ex1.2. 2



$$f(x)=\begin{cases} x & \text{if} \quad \underline{\qquad\qquad\qquad} \\ -1 & \text{if} \quad \underline{\qquad\qquad\qquad} \\ \sqrt{x} & \text{if} \quad \underline{\qquad\qquad\qquad} \end{cases}$$

19.

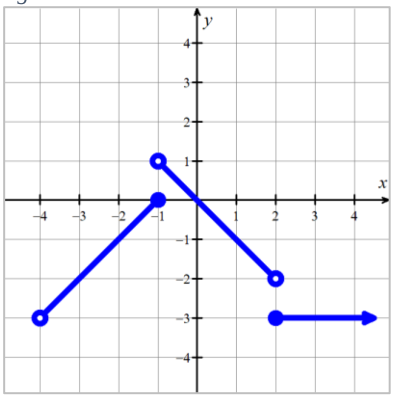
Figure Ex1.2. 3



$$f(x)=\begin{cases} -1 & \text{if} \quad \underline{\qquad\qquad\qquad} \\ x^2 & \text{if} \quad \underline{\qquad\qquad\qquad} \\ 1 & \text{if} \quad \underline{\qquad\qquad\qquad} \end{cases}$$

20.

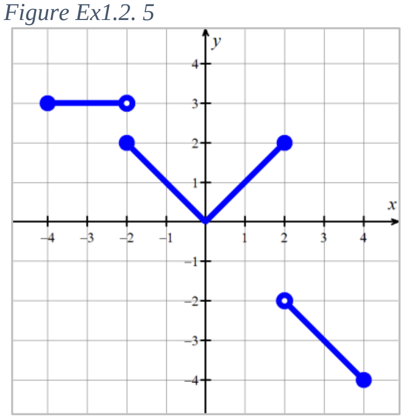
Figure Ex1.2. 4



$$f(x)=\begin{cases} \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad -4 < x \le -1 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad -1 < x < 2 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad x \ge 2 \end{cases}$$

21.

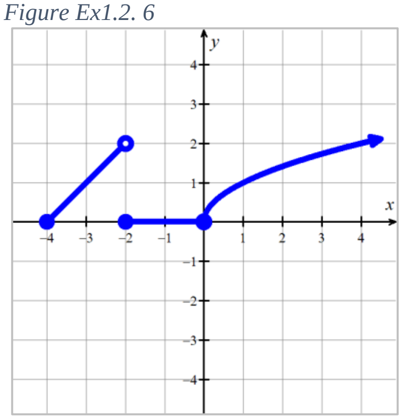
Figure Ex1.2. 5



$$f(x)=\begin{cases} \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad -4 \le x < -2 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad -2 \le x \le 2 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad 2 < x \le 4 \end{cases}$$

22.

Figure Ex1.2. 6

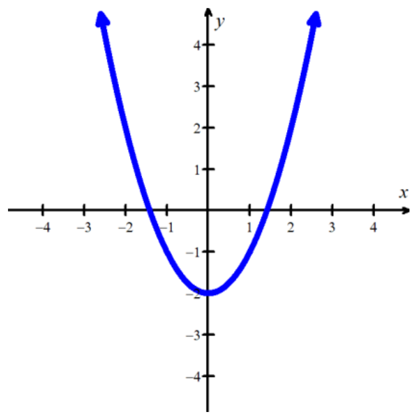


$$f(x)=\begin{cases} \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad -4 \le x < -2 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad -2 \le x < 0 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad x \ge 0 \end{cases}$$

In Exercises 23 – 30, does the graph appear to be symmetric about the *x*-axis, *y*-axis, origin, none of these, or all of these? For graphs representing functions, does the implied symmetry indicate that the function is even, odd, neither, or both?

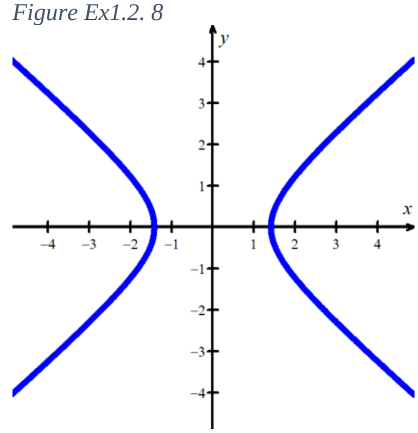
23.

Figure Ex1.2. 7



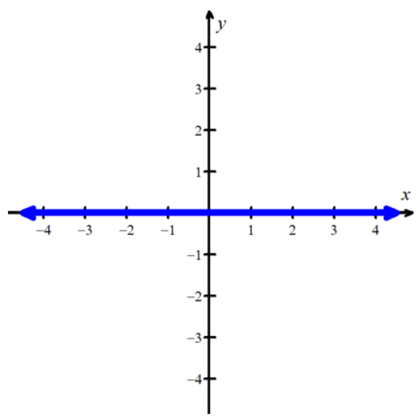
24.

Figure Ex1.2. 8



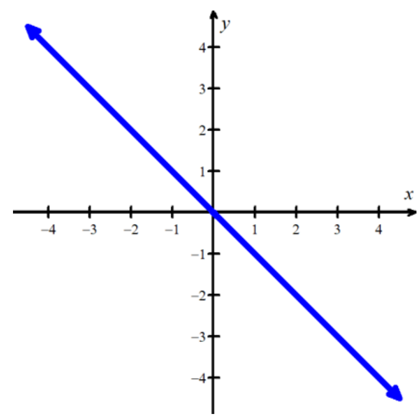
25.

Figure Ex1.2. 9



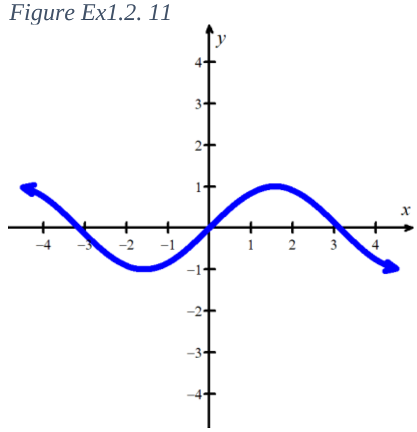
26.

Figure Ex1.2. 10



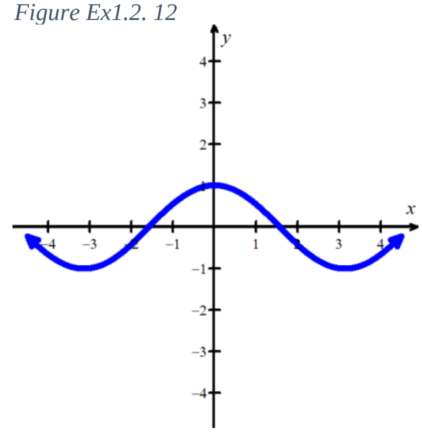
27.

Figure Ex1.2. 11



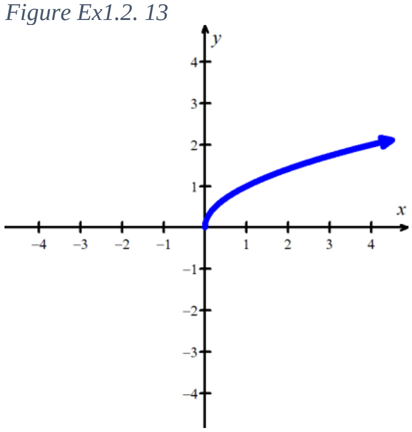
28.

Figure Ex1.2. 12



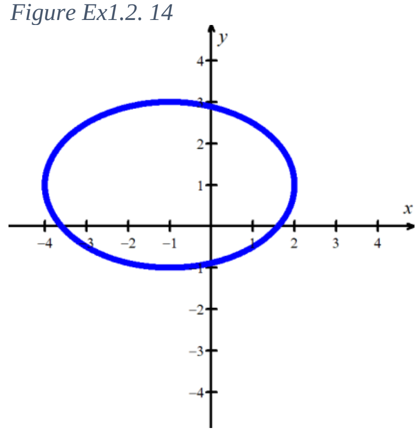
29.

Figure Ex1.2. 13



30.

Figure Ex1.2. 14



In Exercises 31 – 51, determine analytically if the function is even, odd, or neither.

31. $f(x)=7x$

32. $f(x)=7x+2$

33. $f(x)=7$

34. $f(x)=3x^2-4$

35. $f(x)=-x^5+2x^3-x$

36. $f(x)=x^2-x-6$

37. $f(x)=2x^3-x$

38. $f(x)=x^6-x^4+x^2+9$

39. $f(x)=4-x^2$

40. $f(x)=x^3+x^2+x+1$

41. $f(x)=\sqrt{1-x}$

42. $f(x)=\sqrt{1-x^2}$

43. $f(x)=0$

44. $f(x)=\sqrt[3]{x}$

45. $f(x)=\sqrt[3]{x^2}$

46. $f(x)=\dfrac{3}{x^2}$

47. $f(x)=\dfrac{2x-1}{x+1}$

48. $f(x)=\dfrac{3x}{x^2+1}$

49. $f(x)=\dfrac{x^2-3}{x-4x^3}$

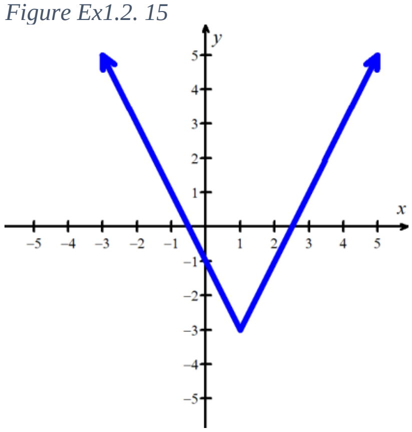
50. $f(x)=\dfrac{9}{\sqrt{4-x^2}}$

51. $f(x)=\dfrac{\sqrt[3]{x^3+x}}{5x}$

In Exercises 52 – 55, use the graph to estimate intervals on which the function is increasing or decreasing.

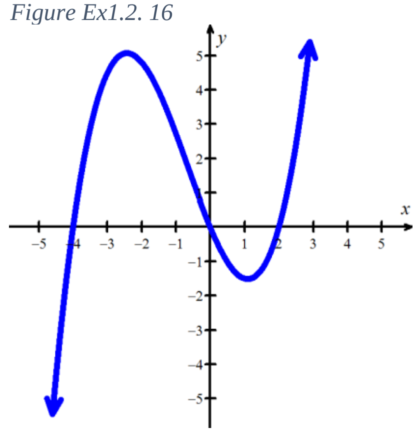
52.

Figure Ex1.2. 15



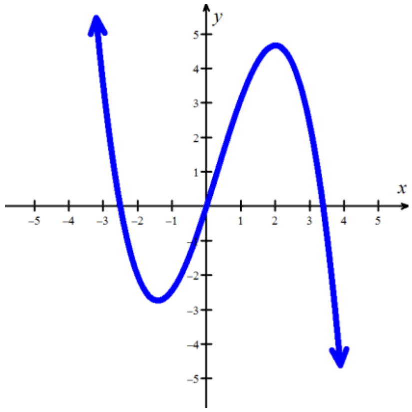
53.

Figure Ex1.2. 16



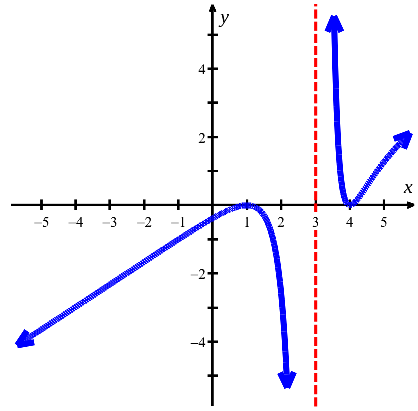
54.

Figure Ex1.2. 17



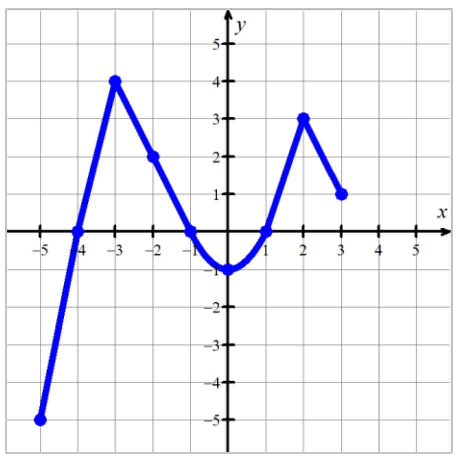
55.

Figure Ex1.2. 18



In Exercises 56 – 71, use the graph of $y = f(x)$ given below to answer the question.

Figure Ex1.2. 19



$y = f(x)$

56. Find the domain of f.

57. Find the range of f.

58. Determine $f(-2)$.

59. Solve $f(x) = 4$.

60. List the x-intercepts, if any exist.

61. List the y-intercept, if any exists.

62. Find the real zeros of f.

63. Solve $f(x) \geq 0$.

64. Find the number of solutions to $f(x) = 1$.

65. Does f appear to be even, odd, or neither?

66. List the intervals on which f is increasing.

67. List the intervals on which f is decreasing.

68. List the local maxima, if any exist.

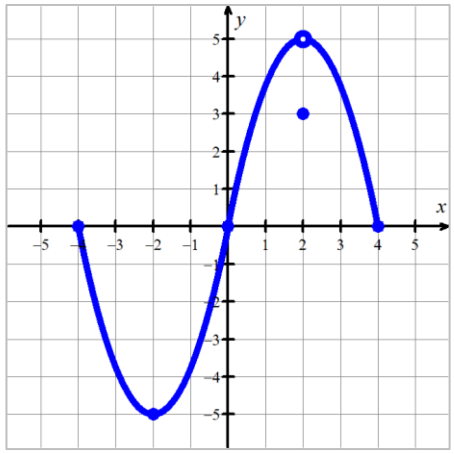
69. List the local minima, if any exist.

70. Find the absolute maximum, if it exists.

71. Find the absolute minimum, if it exists.

In Exercises 72 – 87, use the graph of $y = f(x)$ given below to answer the question.

Figure Ex1.2. 20



$y = f(x)$

72. Find the domain of f.

73. Find the range of f.

74. Determine $f(2)$.

75. Solve $f(x) = -5$.

76. List the *x*-intercepts, if any exist.

77. List the *y*-intercept, if any exists.

78. Find the real zeros of f.

79. Solve $f(x) \leq 0$.

80. Find the number of solutions to $f(x) = 3$.

81. Does f appear to be even, odd, or neither?

82. List the intervals on which f is increasing.

83. List the intervals on which f is decreasing.

84. List the local maxima, if any exist.

85. List the local minima, if any exist.

86. Find the absolute maximum, if it exists.

87. Find the absolute minimum, if it exists.

88. The area A enclosed by a square, in square inches, is a function of the length of one of its sides x, when measured in inches. This relation is expressed by the formula $A(x) = x^2$ for $x > 0$. Find $A(3)$ and solve $A(x) = 36$. Interpret your answers to each. Why is x restricted to $x > 0$?

89. The area A enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This relation is expressed by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

90. The volume V enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides x, when measured in centimeters. This relation is expressed by the formula $V(x) = x^3$ for $x > 0$. Find $V(5)$ and solve $V(x) = 27$. Interpret your answers to each. Why is x restricted to $x > 0$?

91. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

92. The height of an object dropped from the roof of an eight story building is modeled by $h(t) = -16t^2 + 64$, $0 \le t \le 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \le t \le 2$?

93. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \le t \le 12$. Find and interpret $T(0)$, $T(6)$, and $T(12)$.

94. The function $C(x) = x^2 + 10x + 27$ models the cost, in hundreds of dollars, to produce x thousand pens. Find and interpret $C(0)$, $C(2)$, and $C(5)$.

95. Using data from the Bureau of Transportation Statistics, the average fuel economy F, in miles per gallon, for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \le t \le 28$, where t is the number of years since 1980. Use your calculator to find $F(0)$, $F(14)$, and $F(28)$. Round your answers to two decimal places and interpret your answers to each.

96. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \frac{150t}{t+15}$, where t represents the number of years since 1803. Find and interpret $P(0)$ and $P(205)$. Discuss with your classmates what the applied domain and range of P should be.

97. For n copies of the book *Me and my Sasquatch*, a print on demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n)=\begin{cases} 15n & \text{if} \quad 1 \le n \le 25 \\ 13.50n & \text{if} \quad 25 < n \le 50 \\ 12n & \text{if} \quad n > 50 \end{cases}$$

(a) Find and interpret $C(20)$.

(b) How much does it cost to order 50 copies of the book? What about 51 copies?

(c) Your answer to part (b) should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

98. An on-line comic book retailer charges shipping costs according to the formula

$$S(n)=\begin{cases} 1.5n+2.5 & \text{if} \ 1 \le n \le 14 \\ 0 & \text{if} \ n \ge 15 \end{cases}$$

where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

(a) What is the cost to ship 10 comic books?

(b) What is the significance of the formula $S(n)=0$ for $n \ge 15$?

99. The cost C (in dollars) to talk m minutes a month on a mobile phone plan is modeled by

$$C(m)=\begin{cases} 25 & \text{if} \ 0 \le m \le 1000 \\ 25+0.1(m-1000) & \text{if} \ \text{m}>1000 \end{cases}$$

(a) How much does it cost to talk 750 minutes per month with this plan?

(b) How much does it cost to talk 20 hours a month with this plan?

(c) Explain the terms of the plan in words.

100. We define the set of **integers** as $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.[16] The **greatest integer of *x***, denoted by $\lfloor x \rfloor$, is defined to be the largest integer *k* with $k \le x$.

(a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$, and $\lfloor \pi + 6 \rfloor$.

[16] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count'.

(b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piecewise defined function.

HINT: There are infinitely many pieces!

(c) Is $\lfloor a+b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if a or b is an integer? Test some values, make a conjecture, and explain your results.

101. Let $f(x) = \lfloor x \rfloor$ be the greatest integer function as defined in the last exercise.

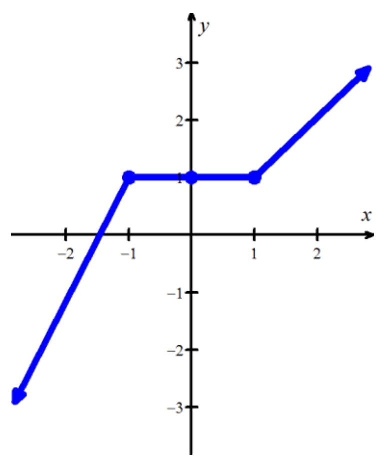
(a) Graph $y = f(x)$. Be careful to correctly describe the behavior of the graph near the integers.

(b) Is f even, odd or neither? Explain.

(c) Discuss with your classmates which points on the graph are local minimums, local maximums, or both. Is f ever increasing? Decreasing? Constant?

102. Consider the graph of the function f given below.

Figure Ex1.2. 21



$y = f(x)$

Refer back to **Definition 1.9** before answering the following.

(a) Show that f has a local maximum, but not a local minimum, at the point $(-1,1)$.

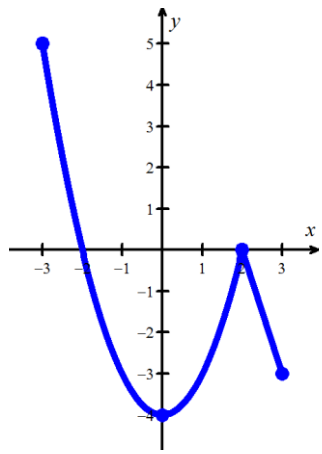
(b) Show that f has a local minimum, but not a local maximum, at the point $(1,1)$.

(c) Show that f has a local maximum AND a local minimum at the point $(0,1)$.

(d) Show that f is constant on the interval $(-1,1)$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

103. Using **Example 1.2.5** as a guide, show that the function g whose graph is given below does not have a local maximum at $(-3,5)$; nor does it have a local minimum at $(3,-3)$. Find its extrema, both local and absolute. What is unique about the point $(0,-4)$ on this graph? Also find the intervals on which g is increasing and the intervals on which g is decreasing.

Figure Ex1.2. 22



$y = g(x)$

1.3 Transformations of Functions

Learning Objectives

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the *x*-axis and the *y*-axis.
- Graph functions using vertical and horizontal scalings.
- Graph functions using a combination of transformations.

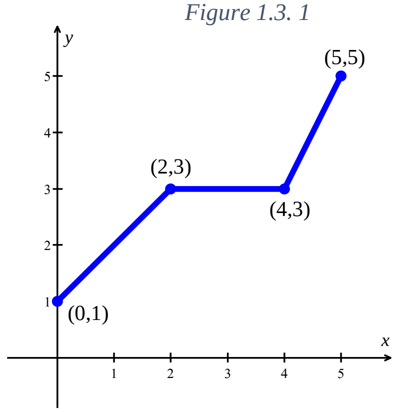
In this section, we study how the graphs of functions change, or **transform**, when certain modifications are made to their inputs or outputs. Transformations fall into two broad categories: **rigid transformations** and **non-rigid transformations**. Rigid transformations, which include **shifts** and **reflections**, do not change the shape of the original graph, only its position and orientation in the plane. Non-rigid transformations, which include scalings, change the shape of the graph. Both types of transformations may affect the domain and/or range of the function.

Adding to, subtracting from, or multiplying the inputs or outputs of a function by a constant are the types of transformations we will discuss here. We first examine making these changes to the output.

Vertical Shifts

Example 1.3.1. Use the function $y = f(x)$, graphed below, to graph the function $y = f(x) + 2$.

Figure 1.3. 1



$y = f(x)$

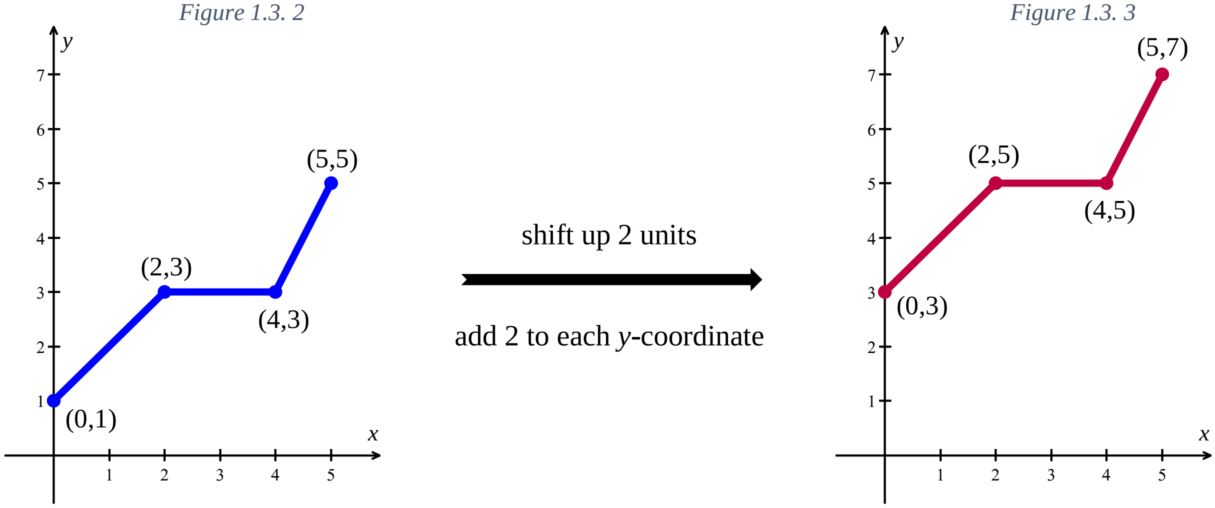
Solution. To evaluate $y = f(x) + 2$ at each *x*-value, we take the output of the original function, $f(x)$, and add 2 to it. The following table shows what's happening.

Input x	Output $y=f(x)$	Coordinates on graph of $y=f(x)$
0	1	$(0,1)$
2	3	$(2,3)$
4	3	$(4,3)$
5	5	$(5,5)$

Input x	Output $y=f(x)+2$	Coordinates on graph of $y=f(x)+2$
0	$1+2=3$	$(0,3)$
2	$3+2=5$	$(2,5)$
4	$3+2=5$	$(4,5)$
5	$5+2=7$	$(5,7)$

The *y*-coordinate of each point on the graph of $y=f(x)+2$ is two more than the *y*-coordinate of $y=f(x)$. Geometrically, adding 2 to the *y*-coordinate of a point moves the point 2 units above its previous location.

Figure 1.3. 2 *Figure 1.3. 3*



$y=f(x)$ $y=f(x)+2$

□

Adding 2 to the *y*-coordinate of every point on a graph is usually described as 'shifting the graph up 2 units'. As seen from the above graphs, this transformation does not affect the domain, but it does affect the range; the range of $y=f(x)$ is $[1,5]$ and the range of $y=f(x)+2$ is $[3,7]$. Since the two graphs have the same shape, this is a rigid transformation.

The same logic explains the transformation that is required to obtain the graph of $y=f(x)-2$ from the graph of $y=f(x)$. Instead of adding 2 to the *y*-coordinates on the graph of $y=f(x)$, we would be subtracting 2. Geometrically, we would be moving the graph down 2 units and the new range would be $[-1,3]$. What we have discussed is generalized below.

Vertical Shift

Suppose f is a function and D is a constant. To graph $y=f(x)+D$, shift the graph of $y=f(x)$ vertically by adding D to the *y*-coordinates of the points on the graph of f. If D is positive, the graph will shift up. If D is negative, the graph will shift down.

A vertical shift is a rigid transformation that only affects the range.

Vertical Reflections

Example 1.3.2. Use the function $y=f(x)$ from **Example 1.3.1** to graph $y=-f(x)$.

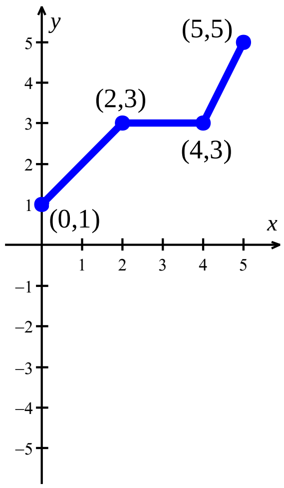
Solution. To evaluate $y=-f(x)$, or $y=(-1)\cdot f(x)$, at each *x*-value, we take the output of the original function, $f(x)$, and multiply it by -1. The following table shows what is happening.

Input x	Output $y=f(x)$	Coordinates on graph of $y=f(x)$
0	1	$(0,1)$
2	3	$(2,3)$
4	3	$(4,3)$
5	5	$(5,5)$

Input x	Output $y=-f(x)$	Coordinates on graph of $y=-f(x)$
0	-1	$(0,-1)$
2	-3	$(2,-3)$
4	-3	$(4,-3)$
5	-5	$(5,-5)$

For every value of x in the domain, $y=-f(x)$ is opposite in sign to $y=f(x)$. Geometrically, multiplying the *y*-coordinate of a point by -1 reflects that point across the *x*-axis.

Figure 1.3. 4

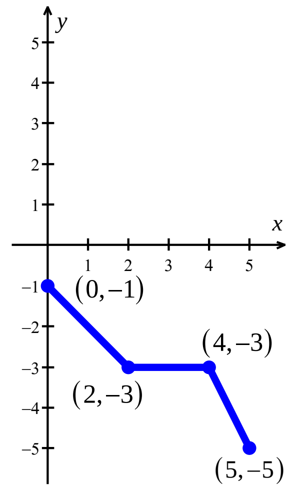


$y = f(x)$

reflect across x-axis

multiply each y-coordinate by -1

Figure 1.3. 5



$y = -f(x)$

□

Multiplying the y-coordinate of every point on a graph by -1 is usually described as 'reflecting the graph across the x-axis'. As seen from the above graphs, this transformation does not affect the domain, but it does affect the range; the range of $y = f(x)$ is $[1,5]$ and the range of $y = -f(x)$ is $[-5,-1]$. The two graphs have the same shape, verifying that this is a rigid transformation.

Vertical Reflections

Suppose f is a function. To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of $y = f(x)$ by -1.

A vertical reflection is a rigid transformation that only affects the range.

Vertical Scalings

Example 1.3.3. Use the function $y = f(x)$ from **Example 1.3.1** to graph $y = 2f(x)$.

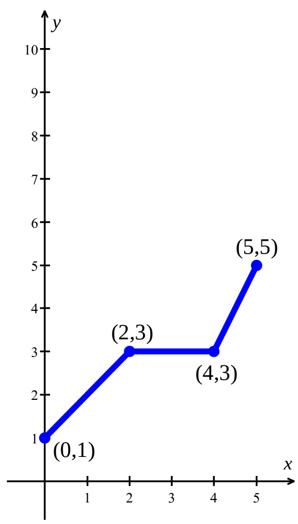
Solution. To evaluate $y = 2f(x)$ at each x-value, we take the output of the original function $f(x)$ and multiply it by 2. The following table shows what is happening.

Input x	Output $y = f(x)$	Coordinates on graph of $y = f(x)$
0	1	$(0,1)$
2	3	$(2,3)$
4	3	$(4,3)$
5	5	$(5,5)$

Input x	Output $y = 2f(x)$	Coordinates on graph of $y = 2f(x)$
0	$2 \cdot 1 = 2$	$(0,2)$
2	$2 \cdot 3 = 6$	$(2,6)$
4	$2 \cdot 3 = 6$	$(4,6)$
5	$2 \cdot 5 = 10$	$(5,10)$

For each value of x in the domain, $y = 2f(x)$ is twice as large as $y = f(x)$. Geometrically, multiplying the *y*-coordinate of a point by the positive number 2 stretches the graph in the *y* direction. This is known as a 'vertical scaling by a factor of 2' and the result is shown below.

Figure 1.3. 6

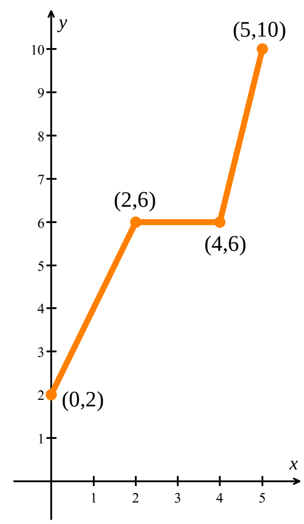


$y = f(x)$

vertical scaling by a factor of 2

multiply each *y*-coordinate by 2

Figure 1.3. 7



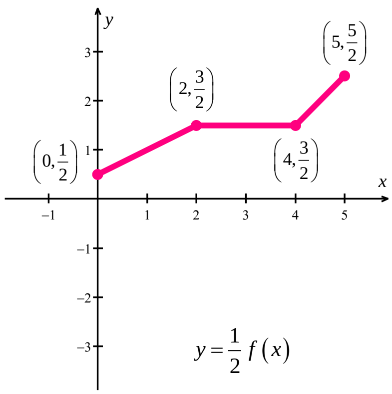
$y = 2f(x)$

□

As seen from the above graphs, this transformation does not affect the domain, but it does affect the range; the range of $y = f(x)$ is $[1,5]$ and the range of $y = 2f(x)$ is $[2,10]$. We note that the two graphs do not have the same shape, so this is a non-rigid transformation.

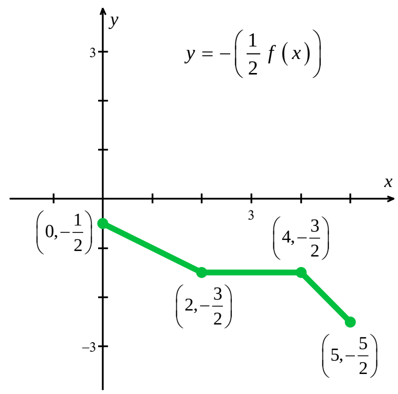
To graph $y = \frac{1}{2}f(x)$, we would multiply all of the *y*-coordinates of the points on the graph of f by $\frac{1}{2}$, resulting in a compression in the *y* direction, or a vertical scaling by a factor of $\frac{1}{2}$. To graph $y = -\frac{1}{2}f(x)$, we multiply all of the *y*-coordinates of the points on the graph of f by $-\frac{1}{2}$. We can rewrite $y=-\frac{1}{2}f(x)$ as $y=-\left(\frac{1}{2}f(x)\right)$. Geometrically, we think of this as two operations: first perform a vertical scaling by a factor of $\frac{1}{2}$ and then reflect the resulting graph across the *x*-axis.

Figure 1.3. 8



vertical scaling by factor of $\frac{1}{2}$

Figure 1.3. 9



reflection across *x*-axis

These results are generalized below.

Vertical Scalings

Suppose f is a function and A is a nonzero constant. To graph $y=Af(x)$, multiply all of the *y*-coordinates of the points on the graph of $y=f(x)$ by A.

- If $A>0$, we say the graph of f has been vertically scaled (stretched if $A>1$ and compressed if $0<A<1$) by a factor of A.
- If $A<0$, the graph of f is both vertically scaled and reflected across the *x*-axis.

A vertical scaling is a non-rigid transformation that only affects the range.

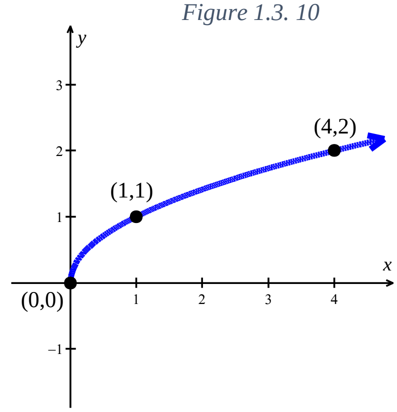
It is sometimes necessary to apply multiple transformations, as in the following example. Here, we limit our transformations to those affecting the output of a function f.

Example 1.3.4. Graph $f(x)=\sqrt{x}$ and plot at least three points. Use transformations to graph $g(x)=2\sqrt{x}-1$. State the domain and range of g.

Solution. Owing to the square root, the domain of $f(x)=\sqrt{x}$ is $x \geq 0$, or $[0,\infty)$. We choose perfect squares to build our table and graph below. [17]

x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$

Figure 1.3. 10



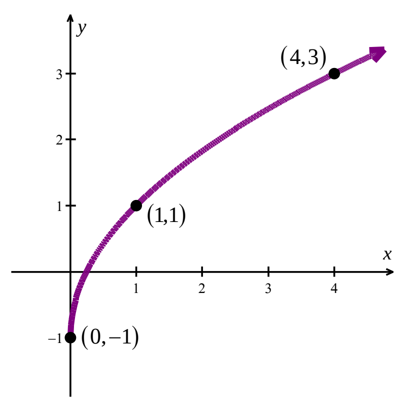
$$y=f(x)=\sqrt{x}$$

From the graph, we can verify that the domain of f is $[0,\infty)$ and the range of f is also $[0,\infty)$.

There are two transformations of f in the function $g(x)=2\sqrt{x}-1$. The order we perform the two transformations in is simply the order of operations in calculating $g(x)=2f(x)-1$. First, we multiply $f(x)=\sqrt{x}$ by 2, and then subtract 1. Multiplication by 2 results in a vertical stretch while subtracting 1 results in a shift down by one unit.

x	$f(x)$	$g(x)=2f(x)-1$	$(x,g(x))$
0	0	$2(0)-1=-1$	$(0,-1)$
1	1	$2(1)-1=1$	$(1,1)$
4	2	$2(2)-1=3$	$(4,3)$

Figure 1.3. 11



$$y=g(x)=2\sqrt{x}-1$$

The domain of g is the same as the domain of f, $[0,\infty)$, but the range of g is $[-1,\infty)$, which is different than the range of f.

□

[17] Recall that we graphed this function earlier as one of the toolkit functions.

Notice that the order of transformations does matter, since $2\left(\sqrt{x}-1\right) \neq 2\sqrt{x}-1$. For correct order of transformations, simply follow the order of arithmetic operations.

Horizontal Shifts

Example 1.3.5. Use the function $y=f\left(x\right)$ from **Example 1.3.1**, to graph $y=f\left(x+2\right)$.

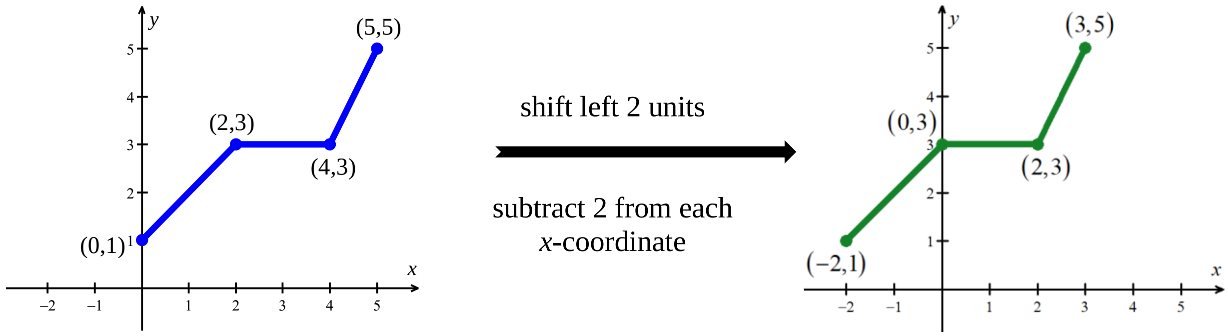
Solution. Both functions, $y=f\left(x\right)$ and $y=f\left(x+2\right)$, have the same range. However, to obtain the same *y*-value we must input different *x*-values. In the following table, to the left, you see several inputs and outputs for $y=f\left(x\right)$, along with their corresponding points on the graph. In the table to the right, for $y=f\left(x+2\right)$, we keep the outputs the same as $y=f\left(x\right)$, but look for new inputs that, when 2 is added, will give us those outputs.

Output $y=f\left(x\right)$	Input x	Coordinates on graph of $y=f\left(x\right)$
1	0	$\left(0,1\right)$
3	2	$\left(2,3\right)$
3	4	$\left(4,3\right)$
5	5	$\left(5,5\right)$

Output $y=f\left(x+2\right)$	Input x	Coordinates on graph of $y=f\left(x+2\right)$
1	need $x+2=0$ so $x=-2$	$\left(-2,1\right)$
3	need $x+2=2$ so $x=0$	$\left(0,3\right)$
3	need $x+2=4$ so $x=2$	$\left(2,3\right)$
5	need $x+2=5$ so $x=3$	$\left(3,5\right)$

The tables show us that, to get $y=f\left(x+2\right)$ to have the same output value as $y=f\left(x\right)$, the input value to $y=f\left(x+2\right)$ must be $x-2$: $y=f\left(\left(x-2\right)+2\right)=f\left(x\right)$. As illustrated in the following side-by-side graphs, this results in a horizontal shift of 2 units to the left.

Figure 1.3. 12 *Figure 1.3. 13*



$y = f(x)$ $y = f(x+2)$

□

As seen from the graphs, this transformation does not affect the range, but it does affect the domain; the domain of $y = f(x)$ is $[0,5]$ while the domain of $y = f(x+2)$ is $[-2,3]$. Since the two graphs have the same shape, this is a rigid transformation.

If we represent a general horizontal shift of $y = f(x)$ by $y = f(x-C)$, then the value of C determines both the amount and direction of the horizontal shift, as shown below.

- In comparing $y = f(x+2)$ with $y = f(x-C)$, we see that $C = -2$. The magnitude of C determines a shift of 2 units, and its sign indicates a shift toward the negative side of the *x*-axis, or to the left.
- Another way to determine C is to set the argument of the function $y = f(x+2)$ equal to zero, giving us $x+2=0$, which we then solve to get $x=-2$. This solution is the value of C.

The same logic explains the needed transformation to the graph of $y = f(x)$ to obtain the graph of $y = f(x-2)$. Instead of subtracting 2 from the *x*-coordinates on the graph of $y = f(x)$, we would be adding 2. Geometrically, we would be moving the graph of f right 2 units and the new domain would be $[2,7]$. What we have discussed is generalized below.

Horizontal Shift

Suppose f is a function and C is a constant. To graph $y = f(x-C)$, shift the graph of $y = f(x)$ horizontally by adding C to the *x*-coordinates of the points on the graph of $y = f(x)$. If C is positive, the graph will shift to the right. If C is negative, the graph will shift to the left.

A horizontal shift is a rigid transformation that only affects the domain.

Horizontal Reflections

Example 1.3.6. Use the function $y = f(x)$ from **Example 1.3.1** to graph $y = f(-x)$.

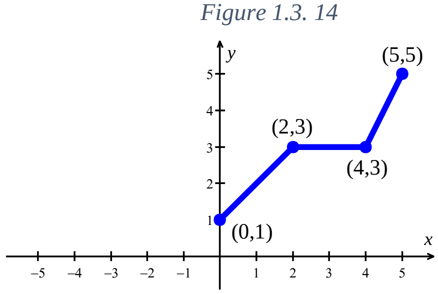
Solution. Both functions, $y = f(x)$ and $y = f(-x)$, have the same range, but to obtain the same *y*-values we must input different *x*-values. In order for $y = f(-x)$ to have the same output as $y = f(x)$, we must input the opposite of each *x*-value: $y = f(-(-x)) = f(x)$. This is demonstrated in the following tables.

Output $y = f(x)$	Input x	Coordinates on graph of $y = f(x)$
1	0	$(0,1)$
3	2	$(2,3)$
3	4	$(4,3)$
5	5	$(5,5)$

Output $y = f(-x)$	Input $-x$	Coordinates on graph of $y = f(-x)$
1	need $-x = 0$ so $x = 0$	$(0,1)$
3	need $-x = 2$ so $x = -2$	$(-2,3)$
3	need $-x = 4$ so $x = -4$	$(-4,3)$
5	need $-x = 5$ so $x = -5$	$(-5,5)$

The tables show us that, to get $y = f(-x)$ to have the same output as $y = f(x)$, the input value to $y = f(-x)$ must be the opposite of the input value to $y = f(x)$. Geometrically, multiplying the *x*-coordinate of a point by -1 reflects the point across the *y*-axis.

Figure 1.3. 14

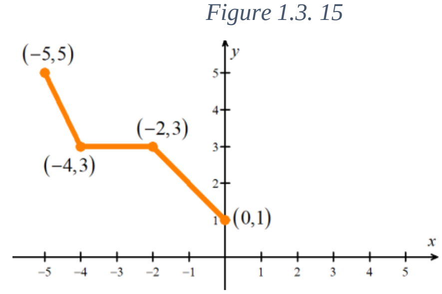


$y = f(x)$

reflect across *y*-axis

multiply each *x*-coordinate by -1

Figure 1.3. 15



$y = f(-x)$

□

Multiplying the *x*-coordinate of every point on a graph by -1 is usually described as 'reflecting the graph across the *y*-axis'. As seen from the graphs, this transformation does not affect the range, but it does affect the domain. The domain of $y = f(x)$ is $[0,5]$ while the domain of $y = f(-x)$ is $[-5,0]$. The two graphs have the same shape, verifying that this is a rigid transformation.

Horizontal Reflections

Suppose f is a function. To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the *y*-axis by multiplying the *x*-coordinates of the points on the graph of $y = f(x)$ by -1.

A horizontal reflection is a rigid transformation that only affects the domain.

Horizontal Scalings

Example 1.3.7. Use the function $y = f(x)$ from **Example 1.3.1** to graph $y = f(2x)$.

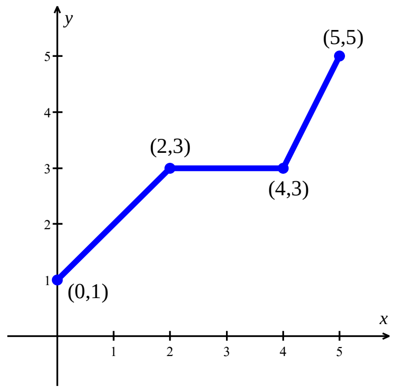
Solution. The functions $y = f(x)$ and $y = f(2x)$ have the same range, but to obtain the same *y*-values we must input different *x*-values. In order for $y = f(2x)$ to have the same output as $y = f(x)$, we must halve the *x*-values: $y = f\left(2\left(\frac{x}{2}\right)\right) = f(x)$. This is demonstrated in the following table.

Output $y = f(x)$	Input x	Coordinates on graph of $y = f(x)$
1	0	$(0,1)$
3	2	$(2,3)$
3	4	$(4,3)$
5	5	$(5,5)$

Output $y = f(2x)$	Input x	Coordinates on graph of $y = f(2x)$
1	need $2x = 0$ so $x = 0$	$(0,1)$
3	need $2x = 2$ so $x = 1$	$(1,3)$
3	need $2x = 4$ so $x = 2$	$(2,3)$
5	need $2x = 5$ so $x = 5/2$	$(5/2,\ 5)$

From the tables, we see that to get $y = f(2x)$ to have the same output as $y = f(x)$, the input value to $y = f(2x)$ must be half of the input value to $y = f(x)$. Geometrically, multiplying the *x*-coordinate of a point by the positive number $\frac{1}{2}$ compresses the graph in the *x* direction.

Figure 1.3. 16



$y = f(x)$

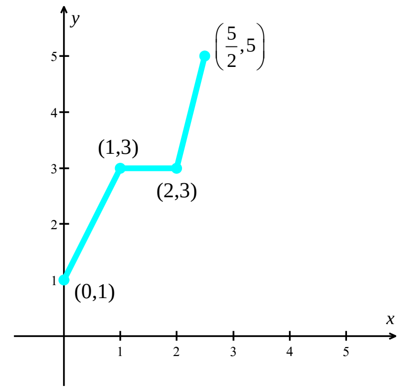
horizontal scaling

by a factor of $\frac{1}{2}$

multiply each

x-coordinate by $\frac{1}{2}$

Figure 1.3. 17



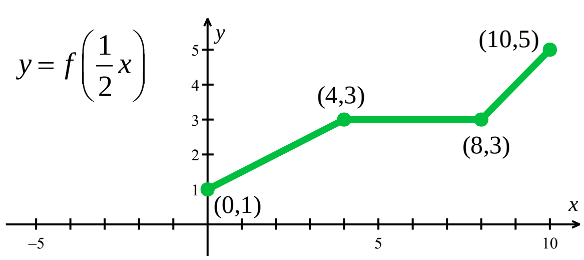
$y = f(2x)$

□

Multiplying the x-coordinate of every point on a graph by $\frac{1}{2}$ is usually described as a 'horizontal scaling by a factor of $\frac{1}{2}$'. As seen from the graphs, this transformation does not affect the range, but it does affect the domain; the domain of $y = f(x)$ is $[0,5]$ while the domain of $y = f(2x)$ is $\left[0,\frac{5}{2}\right]$. Since the two graphs do not have the same shape, this is a non-rigid transformation.

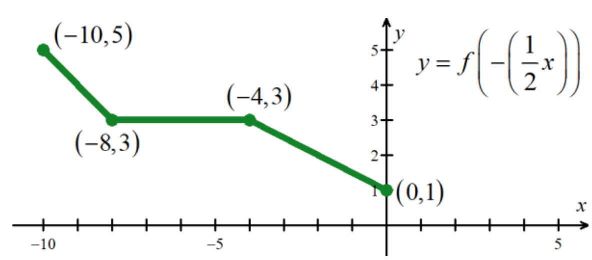
If we wish to graph $y = f\left(\frac{1}{2}x\right)$, we multiply all of the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling by a factor of 2. On the other hand, if we wish to graph $y = f\left(-\frac{1}{2}x\right)$, we multiply all of the x-coordinates of the points on the graph by -2. Geometrically, we can think of two operations. Since $f\left(-\frac{1}{2}x\right) = f\left(-\left(\frac{1}{2}x\right)\right)$, we first perform a horizontal scaling by a factor of 2 and then reflect the resulting graph across the y-axis.

Figure 1.3. 18



horizontal scaling by factor of 2

Figure 1.3. 19



reflection across y-axis

We summarize these results as follows.

Horizontal Scalings

Suppose f is a function and B is a nonzero constant. To graph $y=f(Bx)$, multiply all of the x-coordinates of points on the graph of $y=f(x)$ by $\frac{1}{B}$.

- If $B>0$, we say the graph of f has been horizontally scaled (stretched[18] if $0<B<1$ and compressed[19] if $B>1$) by a factor of $\frac{1}{B}$.
- If $B<0$, the graph of f is both horizontally scaled and reflected across the y-axis.

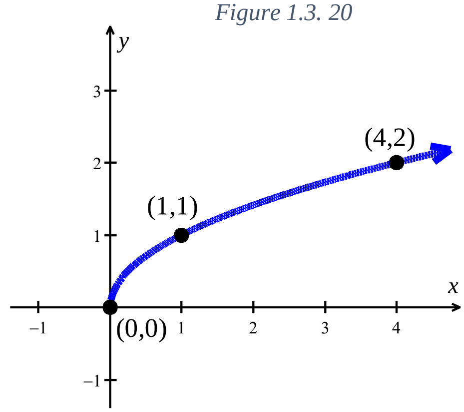
A horizontal scaling is a non-rigid transformation that only affects the domain.

In the next example, we apply multiple transformations to the input of a function.

Example 1.3.8. Use the graph of $f(x)=\sqrt{x}$, shown in **Example 1.3.4**, to sketch the graph of $g(x)=\sqrt{2x+1}$. State the domain and range of g.

Solution. Below is the graph of f.

Figure 1.3. 20



$$y=f(x)=\sqrt{x}$$

There are two transformations of f in the function g. We consider the input value to g that results in the same output value of $y=f(x)$.

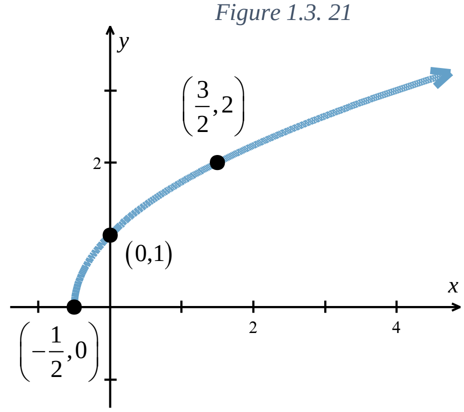
[18] Also called a horizontal expansion.
[19] Also called a horizontal contraction.

x	$y=f(x)=\sqrt{x}$	$(x,f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$

x	$y=g(x)=\sqrt{2x+1}$	$(x,g(x))$
need $2x+1=0$ $\Rightarrow x=-\frac{1}{2}$	0	$\left(-\frac{1}{2},0\right)$
need $2x+1=1$ $\Rightarrow x=0$	1	$(0,1)$
need $2x+1=4$ $\Rightarrow x=\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$

Notice that, in every instance, we first subtract 1 from the *x*-values, resulting in a horizontal shift to the left of 1 unit, then divide the new *x*-values by 2, which is a horizontal scaling. The order we perform the two transformations in is simply the order of operations in calculating the input to g that gives the same output as $y=f(x)$.

Figure 1.3. 21



$y=g(x)=\sqrt{2x+1}$

The domain of g is $\left[-\frac{1}{2},\infty\right)$, while its range is $[0,\infty)$, the same as the range of f.

□

A general technique for finding the required transformations is to consider the input value of $y=g(x)=\sqrt{2x+1}$ that gives the same output as $y=f(x)=\sqrt{x}$. By solving $2\times\mathbf{?}+1=x$, we get

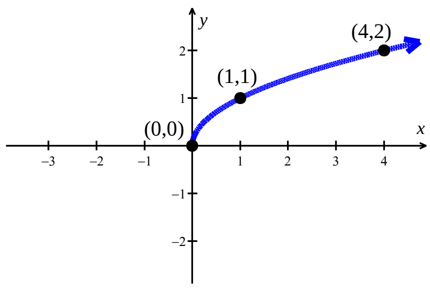
? $= \dfrac{x-1}{2}$ and are thus reminded to follow the order of operations: first subtract 1 from the *x*-value and then divide the result by 2. Notice that the order of transformations does matter since $\dfrac{x}{2}-1 \neq \dfrac{x-1}{2}$.

A Combination of Transformations

Example 1.3.9. Use the graph of $f(x)=\sqrt{x}$, shown in **Example 1.3.4**, to sketch the graph of $g(x)=\sqrt{x+3}-2$. State the domain and range of g.

Solution. To graph the function g, we use two transformations of f, one that affects the input value to f and another that affects the output value from f. For the transformation affecting the input, since $x+3=x-(-3)$, we need to subtract 3 units from each *x*-value, or shift the graph of f left 3 units. The transformation affecting the output requires us to subtract 2 units from each *y*-value, or to shift the graph down 2 units. In either order we will get the same final graph.

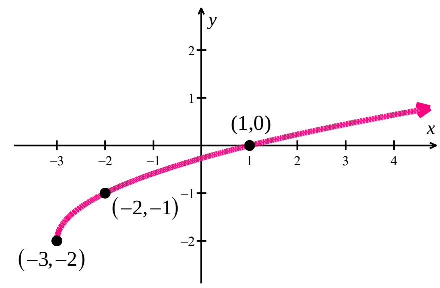
Figure 1.3. 22



$y=f(x)=\sqrt{x}$

shift left 3 units
and down 2 units

Figure 1.3. 23



$y=g(x)=\sqrt{x+3}-2$

We can check our work by finding a specific point on the curve: $g(-3)=\sqrt{-3+3}-2=-2$. The domain of g is $[-3,\infty)$ and its range is $[-2,\infty)$.

□

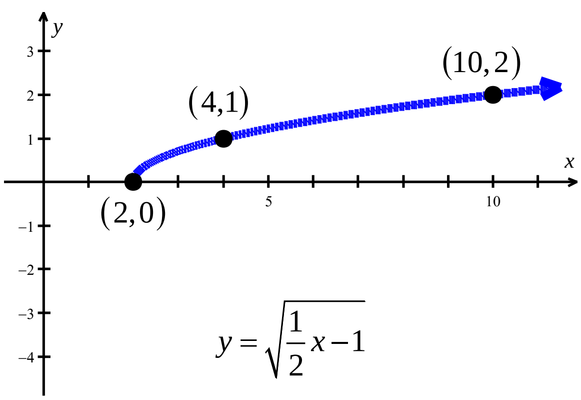
Example 1.3.10. Use the graph of $f(x)=\sqrt{x}$, shown in **Example 1.3.4**, to sketch the graph of $g(x)=-\sqrt{\frac{1}{2}x-1}-2$. State the domain and range of g.

Solution. There are two required transformations affecting the input to f and two required transformations affecting the output from f. We can perform the input transformations first and then the output transformations, or vice-versa.

We choose to begin with the input, and look for the input values to $y=\sqrt{\frac{1}{2}x-1}$ that result in the same output values as $y=f(x)=\sqrt{x}$. Solving $\frac{1}{2}\times\textbf{?}-1=x$ is equivalent to $\textbf{?}=2(x+1)$. Following this order of operations, we first add 1 to each *x*-value, or shift the graph to the right by one unit. We then multiply these new values by 2, which is a horizontal scaling. This results in the graph of $y=\sqrt{\frac{1}{2}x-1}$.

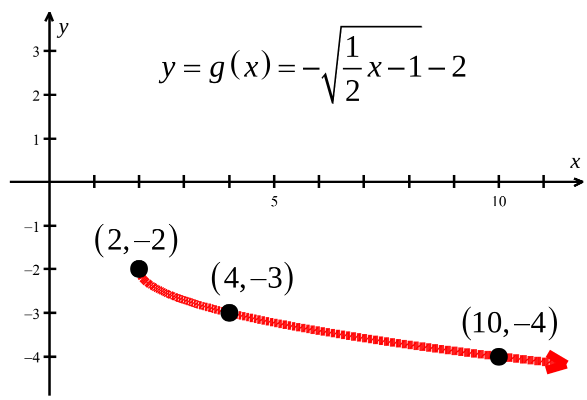
The order of operations for calculating the *y*-values in $y=g(x)=-\sqrt{\frac{1}{2}x-1}-2$ requires first multiplying $y=\sqrt{\frac{1}{2}x-1}$ by −1, which is a vertical reflection, and then subtracting 2 from all *y*-values to shift the graph down 2 units.

Figure 1.3. 24



after horizontal transformations of $y=\sqrt{x}$

Figure 1.3. 25



after vertical transformations of $y=\sqrt{\frac{1}{2}x-1}$

We can check our work by finding a specific point on the curve: $g(2)=-\sqrt{\frac{1}{2}(2)-1}-2=-2$. The domain of g is $[2,\infty)$ and its range is $(-\infty,-2]$.

□

We can apply these ideas in general to obtain the graph of $y=g(x)=A\,f(B\,x-C)+D$ by transforming the graph of $y=f(x)$. We can perform either the input or output transformations first. In the following summary, we begin with the input transformations.

- For the input transformations, consider the input value that results in the same output value as $y=f(x)$. You can think of this as solving $B\times\textbf{?}-C=x$ to get $\textbf{?}=\frac{x+C}{B}$. To go from the input values for f to the input values for g, first add C to the input values for f, and then divide the result by B. The resulting graph is $y = f(Bx - C)$.

- For the output transformations, consider the order of operations for calculating the *y*-values in $y = g(x) = Af(Bx - C) + D$ from $y = f(Bx - C)$: first multiply each *y*-value by A, and then add D.

Graphing Transformations of a Function f

Suppose f is a function. If $A \neq 0$ and $B \neq 0$, then to graph $y = Af(Bx - C) + D$

1. Add C to each *x*-coordinate of the graph of $y = f(x)$.

   This results in a horizontal shift. If C is positive, the graph will shift to the right. If C is negative, the graph will shift to the left.

2. Multiply the *x*-coordinates of the graph obtained in step 1 by $\dfrac{1}{B}$.

   If $B > 0$, this results in a horizontal scaling. If $B < 0$, this results in a horizontal scaling and a reflection about the *y*-axis.

3. Multiply the *y*-coordinates of the graph obtained in step 2 by A.

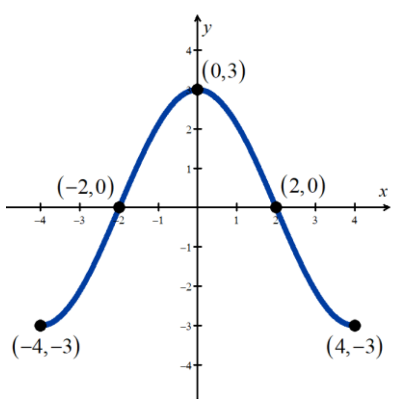
   If $A > 0$, this results in a vertical scaling. If $A < 0$, this results in a vertical scaling and a reflection across the *x*-axis.

4. Add D to the *y*-coordinates of the graph obtained in step 3.

   This results in a vertical shift. If D is positive, the graph will shift up. If D is negative, the graph will shift down.

Example 1.3.11. Below is a complete graph of $y = f(x)$. Use it to graph $g(x) = \dfrac{4 - 3f(1 - 2x)}{2}$.

Figure 1.3. 26

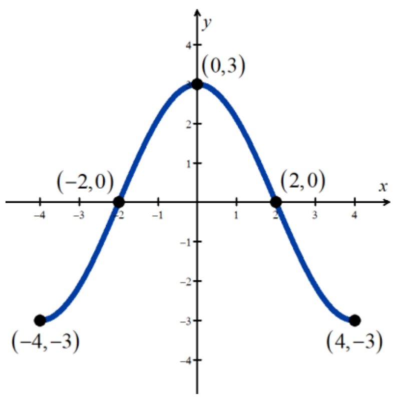


$y = f(x)$

Solution. We first rewrite g in the form $g(x) = -\frac{3}{2} f(-2x+1)+2$, and then follow the four steps as outlined above.

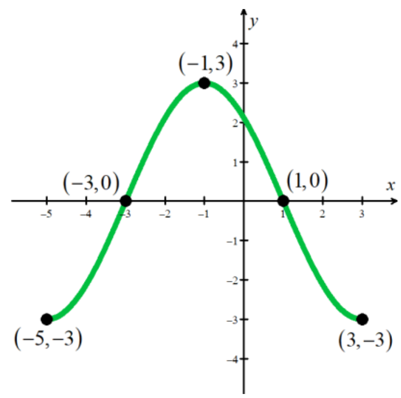
1. Since $C = -1$, our first step is to add -1 to each of the x-coordinates of the points on the graph of $y = f(x)$.

Figure 1.3. 27



$y = f(x)$

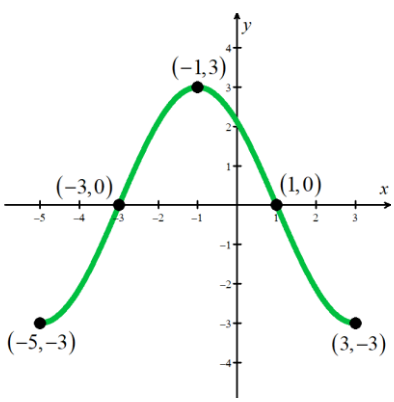
Figure 1.3. 28



$y = f(x+1)$

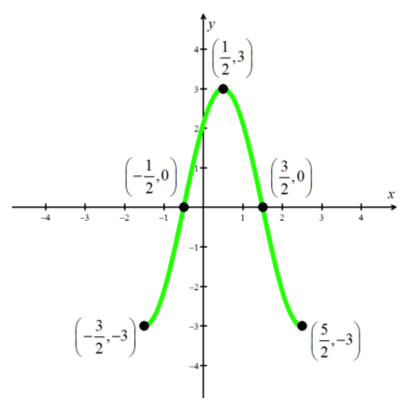
2. With $B = -2$, we next multiply the x-coordinates of the points on the graph of $y = f(x+1)$ by $\frac{1}{-2}$.

Figure 1.3. 29



$y = f(x+1)$

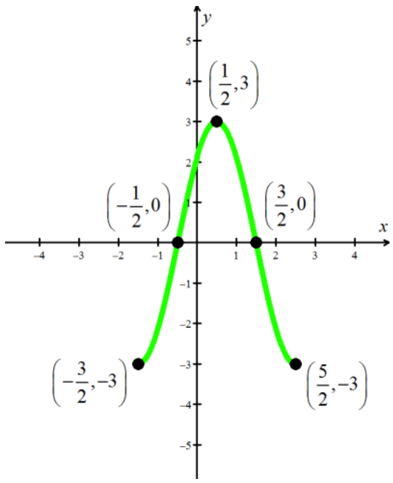
Figure 1.3. 30



$y = f(-2x+1)$

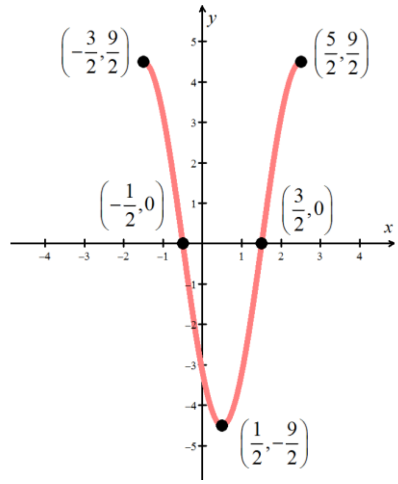
3. Our third step is to multiply each of the *y*-coordinates of the points on the graph of $y = f(-2x+1)$ by A, which is $-\frac{3}{2}$.

Figure 1.3. 31



$y = f(-2x+1)$

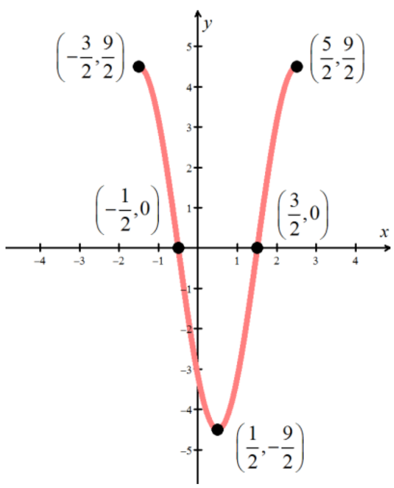
Figure 1.3. 32



$y = -\frac{3}{2} f(-2x+1)$

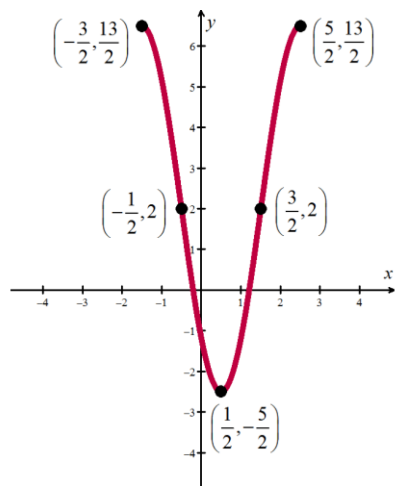
4. In our last step, we add $D = 2$ to each of the *y*-coordinates of the points on the graph of $y = -\frac{3}{2} f(-2x+1)$.

Figure 1.3. 33



$y = -\frac{3}{2} f(-2x+1)$

Figure 1.3. 34



$y = -\frac{3}{2} f(-2x+1) + 2 = g(x)$

□

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations.

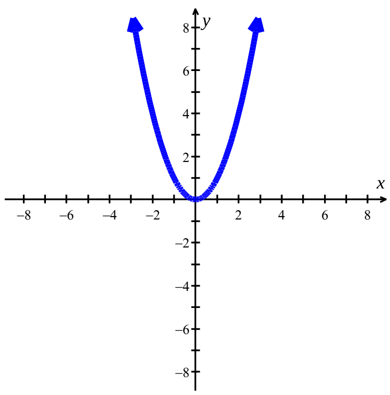
Example 1.3.12. Let $f(x)=x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of f undergoing the following sequence of transformations.

1. Vertical shift up 2 units.
2. Reflection across the *x*-axis.
3. Horizontal shift right 1 unit.
4. Horizontal stretching by a factor of 2.

Solution. We build up to a formula for $g(x)$, beginning with the transformation in part 1.

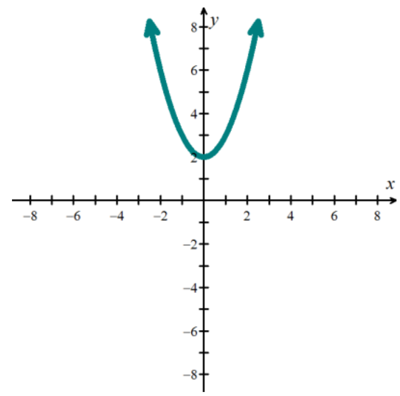
1. To achieve a vertical shift up 2 units, we add 2 to $f(x)=x^2$ and have $y=f(x)+2=x^2+2$.

Figure 1.3. 35



$y=f(x)=x^2$

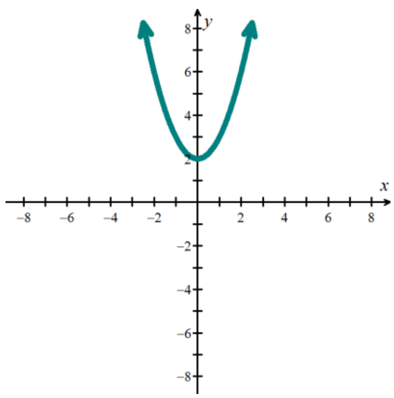
Figure 1.3. 36



$y=f(x)+2=x^2+2$

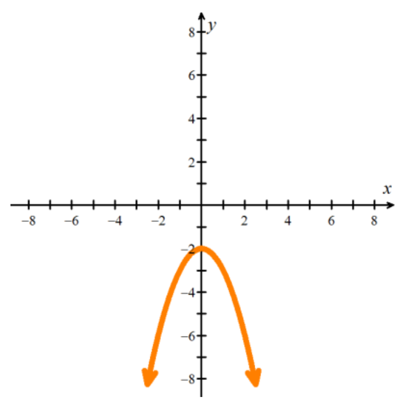
2. Next, we reflect the graph of $y=x^2+2$ about the *x*-axis to get $y=-\left(x^2+2\right)=-x^2-2$.

Figure 1.3. 37



$y=x^2+2$

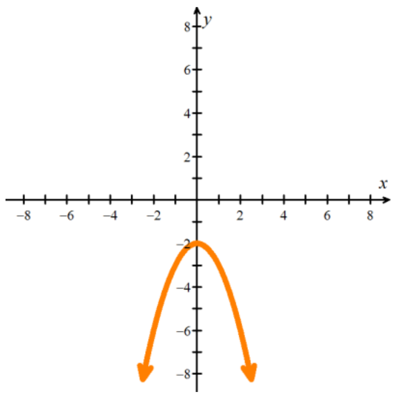
Figure 1.3. 38



$y=-\left(x^2+2\right)=-x^2-2$

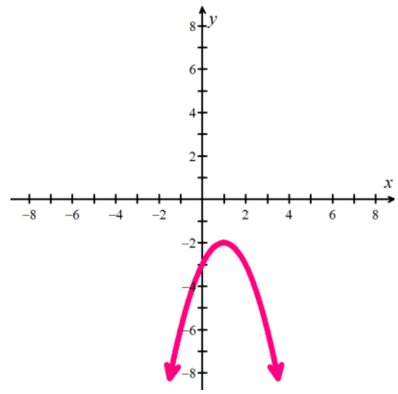
3. The third step is to shift the graph of $y=-x^2-2$ right by one unit, so we get $y=-(x-1)^2-2$.

Figure 1.3. 39



$y=-x^2-2$

Figure 1.3. 40

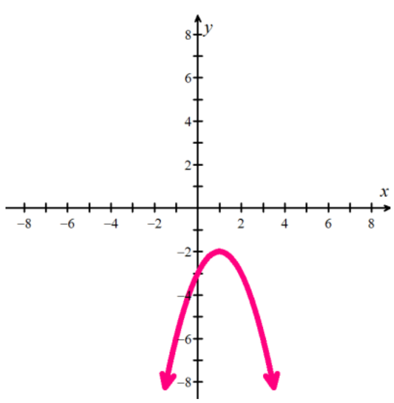


$y=-(x-1)^2-2$

4. Finally, we stretch the graph of $y=-(x-1)^2-2$ horizontally by a factor of 2 as follows:

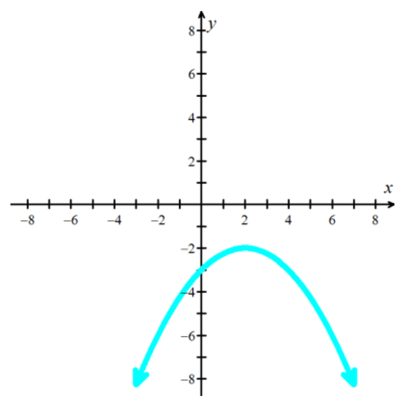
$y=-\left(\left(\frac{1}{2}x\right)-1\right)^2-2$ which, after simplifying, yields $y=-\frac{1}{4}x^2+x-3$. This is the function $g(x)$ that we have been seeking.

Figure 1.3. 41



$y=-(x-1)^2-2$

Figure 1.3. 42



$y=g(x)=-\frac{1}{4}x^2+x-3$

□

1.3 Exercises

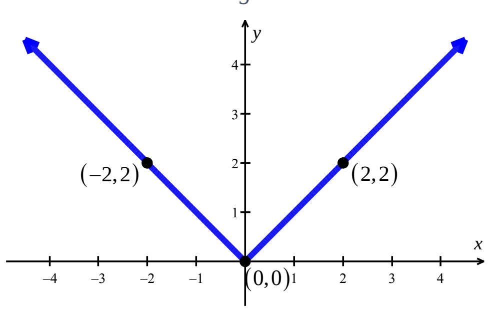
1. When examining the formula of a function that is the result of multiple transformations, how can you distinguish between a horizontal shift and a vertical shift?
2. When examining the formula of a function that is the result of multiple transformations, how can you distinguish between a reflection across the *x*-axis and a reflection across the *y*-axis?

Suppose $(2,-3)$ is on the graph of $y = f(x)$. In Exercises 3 – 20, use the point $(2,-3)$ to find a point on the graph of the given transformed function.

3. $y = f(x)+3$
4. $y = f(x+3)$
5. $y = f(x)-1$
6. $y = f(x-1)$
7. $y = 3f(x)$
8. $y = f(3x)$
9. $y = -f(x)$
10. $y = f(-x)$
11. $y = f(x-3)+1$
12. $y = 2f(x+1)$
13. $y = 10 - f(x)$
14. $y = 3f(2x)-1$
15. $y = \frac{1}{2}f(4-x)$
16. $y = 5f(2x+1)+3$
17. $y = 2f(1-x)-1$
18. $y = f\left(\frac{7-2x}{4}\right)$
19. $y = \frac{f(3x)-1}{2}$
20. $y = \frac{4-f(3x-1)}{7}$

The complete graph of $y = f(x)$ is given below. In Exercises 21 – 29, use it to sketch a graph of the given transformed function.

Figure Ex1.3. 1



The graph of $y = f(x)$ for Exercises 21 – 29

21. $y = f(x)+1$
22. $y = f(x)-2$
23. $y = f(x+1)$
24. $y = f(x-2)$
25. $y = 2f(x)$
26. $y = f(2x)$

27. $y = 2 - f(x)$

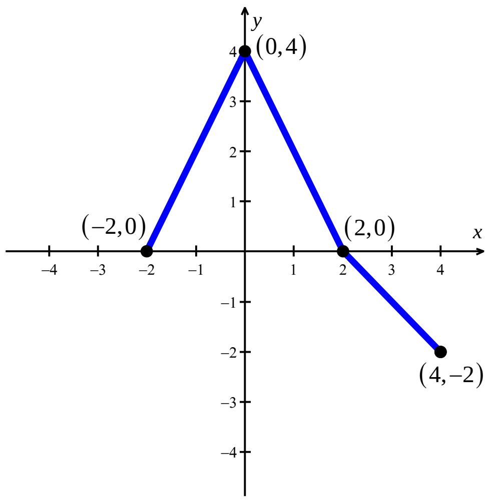
28. $y = f(2 - x)$

29. $y = 2 - f(2 - x)$

30. Some of the answers to Exercises 21 – 29 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?

The complete graph of $y = f(x)$ is given below. In Exercises 31 – 39, use it to sketch a graph of the given transformed function.

Figure Ex1.3. 2



The graph of $y = f(x)$ for Exercises 31 – 39

31. $y = f(x) - 1$

32. $y = f(x + 1)$

33. $y = \frac{1}{2} f(x)$

34. $y = f(2x)$

35. $y = -f(x)$

36. $y = f(-x)$

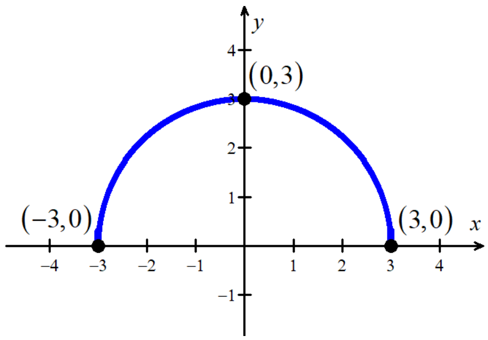
37. $y = f(x + 1) - 1$

38. $y = 1 - f(x)$

39. $y = \frac{1}{2} f(x + 1) - 1$

The complete graph of $y = f(x)$ is given below. In Exercises 40 – 51, use it to sketch a graph of the given transformed function.

Figure Ex1.3. 3



The graph of $y = f(x)$ for Exercises 40 – 51

40. $g(x) = f(x) + 3$

41. $h(x) = f(x) - \frac{1}{2}$

42. $j(x) = f\left(x - \frac{2}{3}\right)$

43. $a(x) = f(x+4)$

44. $b(x) = f(x+1) - 1$

45. $c(x) = \frac{3}{5} f(x)$

46. $d(x) = -2f(x)$

47. $k(x) = f\left(\frac{2}{3}x\right)$

48. $m(x) = -\frac{1}{4} f(3x)$

49. $n(x) = 4f(x-3) - 6$

50. $p(x) = 4 + f(1-2x)$

51. $q(x) = -\frac{1}{2} f\left(\frac{x+4}{2}\right) - 3$

52. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift right 2 units; (2) shift down 3 units.

53. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift down 3 units; (2) shift right 2 units.

54. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) reflect across the x-axis; (2) shift up 1 unit.

55. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift up 1 unit; (2) reflect across the x-axis.

56. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units.

57. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units.

58. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift left 3 units; (2) scale vertically by a factor of 2; (3) shift down 4 units.

59. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift left 3 units; (2) shift down 4 units; (3) scale vertically by a factor of 2.

60. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift right 3 units; (2) scale horizontally by a factor of $\frac{1}{2}$; (3) shift up 1 unit.

61. Write a formula for a function g whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) scale horizontally by a factor of $\frac{1}{2}$; (2) shift right 3 units; (3) shift up 1 unit.

62. Write a formula for a function g whose graph is obtained from $f(x) = |x|$ after the sequence of transformations: (1) shift down 3 units; (2) shift right 1 unit.

63. Write a formula for a function g whose graph is obtained from $f(x) = \frac{1}{x}$ after the sequence of transformations: (1) shift down 4 units; (2) shift right 3 units.

64. Write a formula for a function g whose graph is obtained from $f(x) = \frac{1}{x^2}$ after the sequence of transformations: (1) shift up 2 units; (2) shift left 4 units.

65. Write a formula for a function g whose graph is obtained from $f(x) = |x|$ after the sequence of transformations: (1) reflect across the y-axis; (2) scale horizontally by a factor of $\frac{1}{4}$.

66. Write a formula for a function g whose graph is obtained from $f(x) = \frac{1}{x^2}$ after the sequence of transformations: (1) scale vertically by a factor of $\frac{1}{3}$; (2) shift left 2 units; (3) shift down 3 units.

67. Write a formula for a function g whose graph is obtained from $f(x) = \frac{1}{x}$ after the sequence of transformations: (1) scale vertically by a factor of 8; (2) shift right 4 units; (3) shift up 2 units.

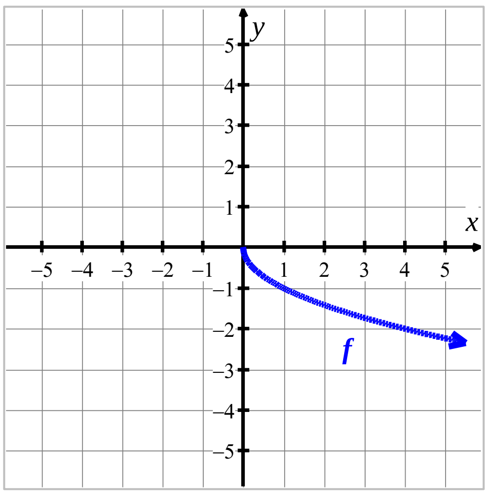
68. Write a formula for a function g whose graph is obtained from $f(x) = x^2$ after the sequence of transformations: (1) scale vertically by a factor of $\frac{1}{2}$; (2) shift right 5 units; (3) shift up 1 unit.

69. Write a formula for a function g whose graph is obtained from $f(x) = x^2$ after the sequence of transformations: (1) scale horizontally by a factor of $\frac{1}{3}$; (2) shift left 4 units; (3) shift down 3 units.

In Exercises 70 – 79, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

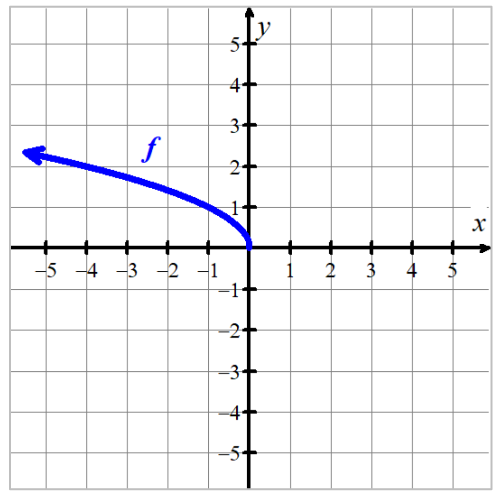
70.

Figure Ex1.3. 4



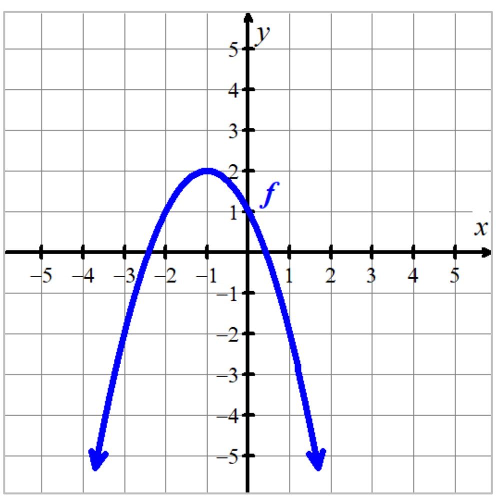
71.

Figure Ex1.3. 5



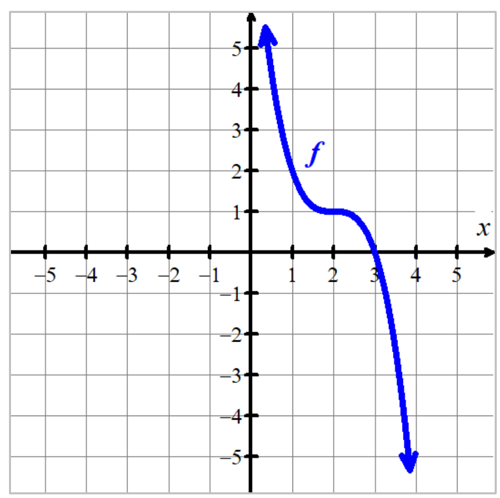
72.

Figure Ex1.3. 6



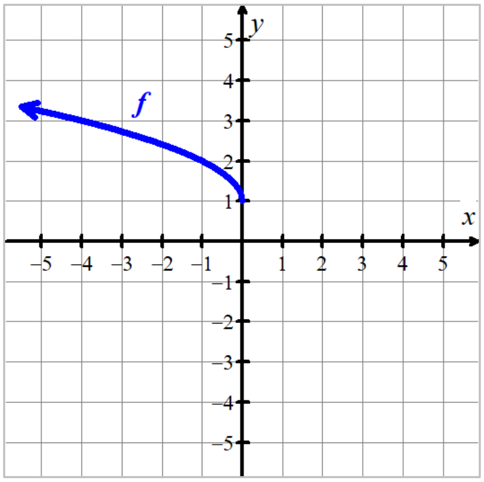
73.

Figure Ex1.3. 7



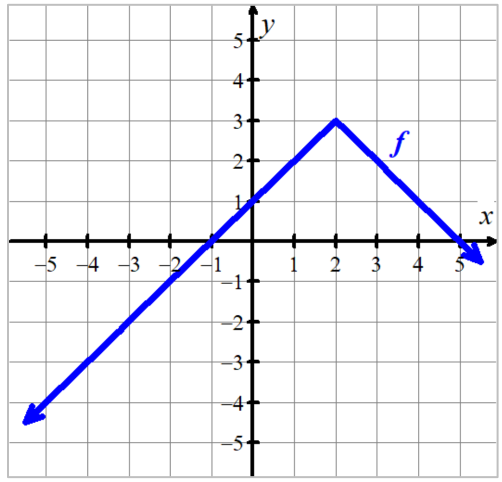
74.

Figure Ex1.3. 8



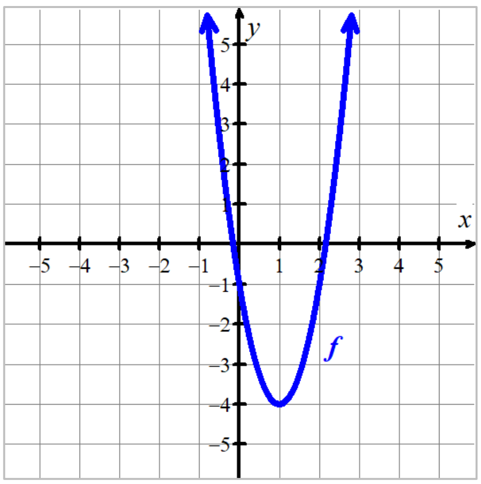
75.

Figure Ex1.3. 9



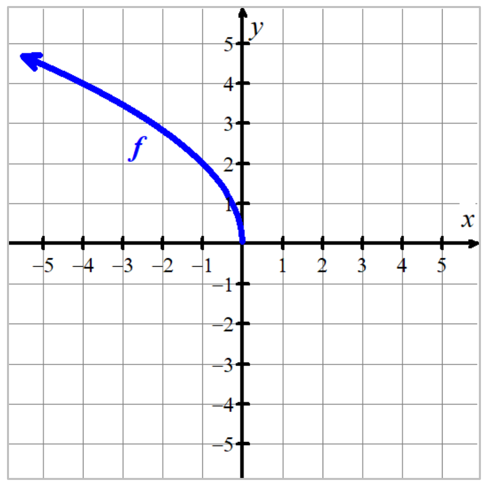
76.

Figure Ex1.3. 10



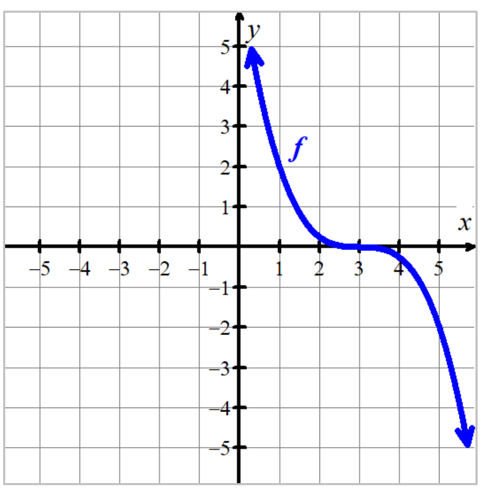
77.

Figure Ex1.3. 11



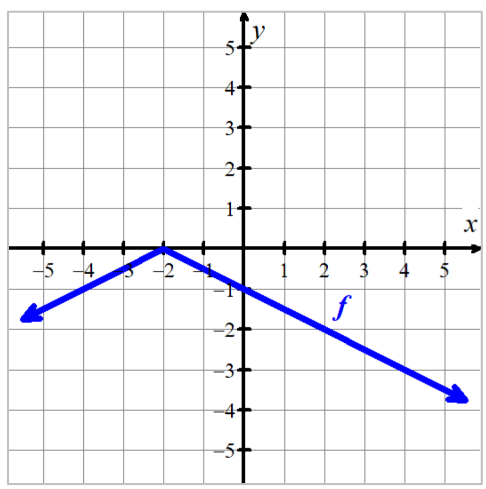
78.

Figure Ex1.3. 12



79.

Figure Ex1.3. 13



In Exercises 80 – 92, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

80. $f(x) = (x+1)^2 - 3$

81. $h(x) = |x-1| + 4$

82. $k(x) = (x-2)^3 - 1$

83. $m(x) = 3 + \sqrt{x+2}$

84. $g(x) = 4(x+1)^2 - 5$

85. $g(x) = 5(x+3)^2 - 2$

86. $h(x) = -2|x-4| + 3$

87. $k(x) = -3\sqrt{x} - 1$

88. $m(x) = \frac{1}{2}x^3$

89. $n(x) = \frac{1}{3}|x-2|$

90. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$

91. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$

92. $a(x) = \sqrt{-x+4}$

93. For many common functions, the properties of algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that $\sqrt{9x} = 3\sqrt{x}$. With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$. What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

94. As mentioned earlier in the section, in general, the order in which transformations are applied matters. Yet, in one of our examples with two transformations, the order did not matter. With the help of your classmates, determine the situations in which order does matter and those in which it does not.

95. What happens if you reflect an even function across the *y*-axis?

96. What happens if you reflect an odd function across the *y*-axis?

97. What happens if you reflect an even function across the *x*-axis?

98. What happens if you reflect an odd function across the *x*-axis?

99. How would you describe symmetry about the origin in terms of reflections?

1.4. Combinations of Functions

Learning Objectives

- Find and simplify functions involving arithmetic expressions.
- Combine functions through addition, subtraction, multiplication, and division.
- Determine the domain of a function resulting from an arithmetic operation.
- Find the difference quotient of a function.
- Create a new function through composition of functions.
- Find the domain of a composite function.
- Find values of composite functions.
- Decompose a composite function into its component functions.

We begin this section by again evaluating functions, but now add expressions to the numerical values we have focused on up to this point. We then move on to combining functions using the four basic arithmetic operations, and later introduce function composition, providing us with yet another way to combine functions.

Functions Involving Arithmetic Expressions

Through the following example, we begin finding and simplifying functions of expressions and expressions of functions, in preparation for arithmetic operations involving functions.

Example 1.4.1. Let $f(x) = -x^2 + 3x + 4$. Find and simplify the following.

1. $f(2x)$, $2f(x)$
2. $f(x+2)$, $f(x)+2$, $f(x)+f(2)$

Solution.

1. To find $f(2x)$, we replace every occurrence of x with the expression $2x$.

$$\begin{aligned} f(x) &= -x^2 + 3x + 4 \\ f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply $f(x)$ by 2.

$$\begin{aligned} 2f(x) &= 2\left(-x^2+3x+4\right) \\ &= -2x^2+6x+8 \end{aligned}$$

2. To find $f(x+2)$, we replace every occurrence of x with the expression $x+2$.

$$\begin{aligned} f(x) &= -x^2+3x+4 \\ f(x+2) &= -(x+2)^2+3(x+2)+4 \\ &= -\left(x^2+4x+4\right)+(3x+6)+4 \\ &= -x^2-4x-4+3x+6+4 \\ &= -x^2-x+6 \end{aligned}$$

To find $f(x)+2$, we add 2 to $f(x)$.

$$\begin{aligned} f(x)+2 &= \left(-x^2+3x+4\right)+2 \\ &= -x^2+3x+6 \end{aligned}$$

For $f(x)+f(2)$, we evaluate $f(x)$ and $f(2)$ separately and then add the results.

$$\begin{aligned} f(x)+f(2) &= \left(-x^2+3x+4\right)+\left(-(2)^2+3(2)+4\right) \\ &= -x^2+3x+4+6 \\ &= -x^2+3x+10 \end{aligned}$$

□

A couple notes about **Example 1.4.1** are in order.

1. First, note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the input by 2; for $2f(x)$, we are multiplying the output by 2. As we see, we get entirely different results.
2. Also note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three different expressions. While $f(x+2)$ is the function f evaluated at $(x+2)$, $f(x)+2$ is the addition of 2 to each function value $f(x)$, and $f(x)+f(2)$ is the sum of the function values $f(x)$ and $f(2)$.
3. Observe the use of parentheses when substituting one algebraic expression into another, an important practice in evaluating functions.

Arithmetic Operations

We have used function notation to make sense of expressions such as $f(x)+2$ and $2f(x)$ for a given function f. It would seem natural then that functions should have their own arithmetic that is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply, and divide functions using the arithmetic we already know for real numbers.

Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.[20]

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula

$$(f+g)(x)=f(x)+g(x)$$

- The **difference** of f and g, denoted $f-g$, is the function defined by the formula

$$(f-g)(x)=f(x)-g(x)$$

- The **product** of f and g, denoted $f\cdot g$, is the function defined by the formula

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, \text{ provided } g(x)\neq 0.$$

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that the product of two functions may be written without the product sign; that is, $(fg)(x)=(f\cdot g)(x)$.

Example 1.4.2. Let $f(x)=6x^2-2x$ and $g(x)=3-\dfrac{1}{x}$. Find the following.

1. $(f+g)(-1)$
2. $(f\cdot g)(2)$

Solution.

1. To find $(f+g)(-1)$ we first find $f(-1)=8$ and $g(-1)=4$. By definition, we have

$$\begin{aligned}(f+g)(-1)&=f(-1)+g(-1)\\&=8+4\\&=12\end{aligned}$$

[20] Thus x is an element of the intersection of the two domains.

2. For $(f \cdot g)(2)$, we need $f(2)$ and $g(2)$. Since $f(2)=20$ and $g(2)=\frac{5}{2}$, by definition, we have

$$\begin{aligned}(f \cdot g)(2) &= f(2)\cdot g(2)\\ &=(20)\left(\frac{5}{2}\right)\\ &=50\end{aligned}$$

□

Note that in the previous example, an alternate method is to first find and simplify $(f+g)(x)$ and $(f \cdot g)(x)$, and then evaluate these simplified expressions at -1 and 2, respectively.

Example 1.4.3. As in **Example 1.4.2**, let $f(x)=6x^2-2x$ and $g(x)=3-\frac{1}{x}$. Find and simplify $(g-f)(x)$. Determine the domain of the resulting function.

Solution. To find $(g-f)(x)$, we begin with the definition and proceed with simplifying the resulting formula.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &=\left(3-\frac{1}{x}\right)-\left(6x^2-2x\right)\\ &=3-\frac{1}{x}-6x^2+2x\\ &=\frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x} \quad \text{obtain common denominator}\\ &=\frac{3x-1-6x^3+2x^2}{x}\\ &=\frac{-6x^3+2x^2+3x-1}{x}\end{aligned}$$

To find the domain of $g-f$, we find the domains of g and f separately, and then determine the intersection of these two sets. We will demonstrate later why we should not use the resulting function to find the domain. Owing to the denominator in the expression $g(x)=3-\frac{1}{x}$, we get that the domain of g is $(-\infty,0)\cup(0,\infty)$. Since $f(x)=6x^2-2x$ is valid for all real numbers, we have no further restrictions. Thus, the domain of $g-f$ matches the domain of g, namely $(-\infty,0)\cup(0,\infty)$.

□

Example 1.4.4. As in the previous two examples, let $f(x)=6x^2-2x$ and $g(x)=3-\frac{1}{x}$. Find and simplify $\left(\frac{g}{f}\right)(x)$. Determine the domain of the resulting function.

Solution. We begin with the definition for the quotient, noting that we are finding $\left(\frac{g}{f}\right)(x)$ instead of $\left(\frac{f}{g}\right)(x)$, and we simplify the resulting expression.

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{3-\frac{1}{x}}{6x^2-2x} \\
&= \frac{3-\frac{1}{x}}{6x^2-2x}\cdot\frac{x}{x} && \text{simplify complex fraction} \\
&= \frac{3x-1}{6x^3-2x^2} \\
&= \frac{3x-1}{\left(2x^2\right)(3x-1)} && \text{factor} \\
&= \frac{1}{2x^2} && \text{divide out the common factor } 3x-1
\end{aligned}$$

To find the domain of $\frac{g}{f}$, we start by identifying the domains of f and g separately. In **Example 1.4.3**, we found that the domain of g is $(-\infty,0)\cup(0,\infty)$ and the domain of f is $(-\infty,\infty)$. Thus, as in **Example 1.4.3**, we exclude $x=0$ from the domain. Additionally, with a quotient of functions, we must guard against the denominator being 0. In this case, for $\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}$, we must guarantee that $f(x)$ is not equal to 0. Setting $f(x)=0$ gives $6x^2-2x=0$, which occurs when $x=0$ or $x=\frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x=0$ and $x=\frac{1}{3}$, or $(-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$.

□

Please note the importance of finding the domain of a function before simplifying its expression. In **Example 1.4.4**, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we would only have the expression $\frac{1}{2x^2}$ to go by and we would (incorrectly!) state the domain as $(-\infty,0)\cup(0,\infty)$, since the other troublesome number, $x=\frac{1}{3}$, was canceled away.[21]

Next, we turn our attention to the **difference quotient** of a function.

The Difference Quotient

Definition 1.10. Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h)-f(x)}{h}$$

For reasons that will become clear in Calculus, simplifying a difference quotient is an important skill. Graphically, we may interpret the difference quotient as being the slope of a line connecting two points on a curve. For the following graph of $y=f(x)$, the slope of the line connecting the points $(x, f(x))$ and $(x+h, f(x+h))$ can be found by taking 'rise over run', which is $\frac{f(x+h)-f(x)}{h}$.

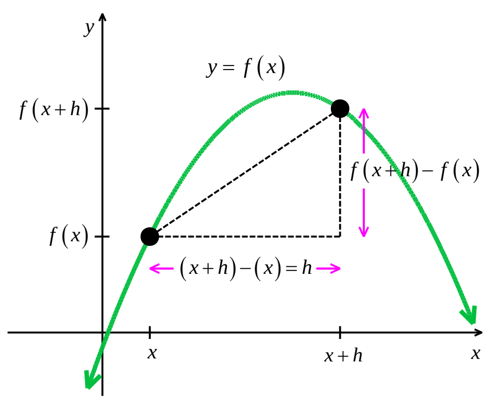


Figure 1.4. 1

Example 1.4.5. Find and simplify the difference quotients for the following functions.

1. $f(x)=x^2-x-2$
2. $g(x)=\frac{3}{2x+1}$

Solution.

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x)=x^2-x-2$ with the expression $(x+h)$ to get

[21] We'll see what this means geometrically in **Chapter 3**.

$$\begin{aligned} f(x) &= x^2 - x - 2 \\ f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2 \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h(2x + h - 1)}{h} \\ &= 2x + h - 1 \end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \frac{3}{2x+1}$ with the expression $(x+h)$ to get

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x + 2h + 1} \end{aligned}$$

This yields

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \\ &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\ &= \frac{3(2x+1) - 3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\ &= \frac{6x + 3 - 6x - 6h - 3}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6h}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6}{(2x+2h+1)(2x+1)} \end{aligned}$$

□

We next introduce composition of functions as a method for combining functions.

Function Composition

Definition 1.11. Suppose f and g are two functions. The **composite** of g with f, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided x is in the domain of f and $f(x)$ is in the domain of g.

The quantity $g \circ f$ is also read 'g composed with f' or, more simply, 'g of f'. At its most basic level, **Definition 1.11** tells us that to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of x in the formula for $g(x)$ with the formula we have for $f(x)$.

If we take a step back and look at this from a procedural 'inputs and outputs' perspective, **Definifion 1.11** tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f, then applying the procedure g, as illustrated below.

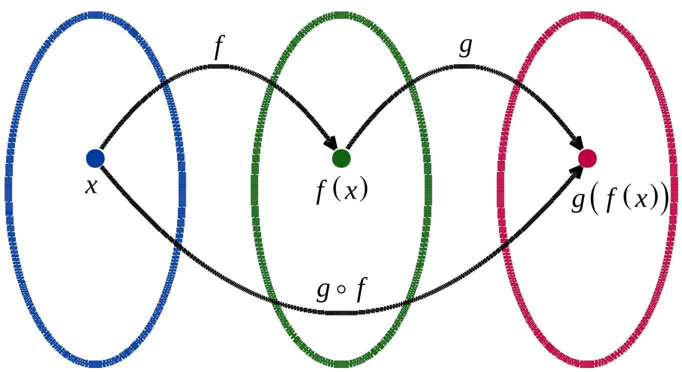


Figure 1.4. 2

In the expression $g(f(x))$, the function f is often called the **inside function** while g is called the **outside function**. We proceed with an example in which we determine formulas for composite functions from three given functions: f, g, and h.

Example 1.4.6. Let $f(x) = 2x+3$, $g(x) = \dfrac{1}{x-1}$, and $h(x) = \sqrt{x+4}$. Find and simplify expressions for the following functions.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(h \circ f)(x)$
4. $(f \circ f)(x)$
5. $(g \circ g)(x)$
6. $(h \circ g \circ f)(x)$

Solution.

1. By **Definition 1.11**, $(g \circ f)(x) = g(f(x))$, and we have

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x+3) && \text{since } f(x) = 2x+3 \\ &= \frac{1}{(2x+3)-1} && \text{substitute } (2x+3) \text{ for } x \text{ in } g(x) = \frac{1}{x-1} \\ &= \frac{1}{2x+2} \end{aligned}$$

2. After swapping the g and f in part 1, we find an expression for $(f \circ g)(x)$.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{by definition} \\ &= f\left(\frac{1}{x-1}\right) && \text{since } g(x) = \frac{1}{x-1} \\ &= 2\left(\frac{1}{x-1}\right)+3 && \text{substitute } \left(\frac{1}{x-1}\right) \text{ for } x \text{ in } f(x) = 2x+3 \\ &= \frac{2}{x-1} + 3 \cdot \frac{x-1}{x-1} && \text{obtain common denominator} \\ &= \frac{3x-1}{x-1} \end{aligned}$$

3. We next find an expression for $(h \circ f)(x)$.

$$\begin{aligned} (h \circ f)(x) &= h(f(x)) && \text{by definition} \\ &= h(2x+3) && \text{since } f(x) = 2x+3 \\ &= \sqrt{(2x+3)+4} && \text{substitute } (2x+3) \text{ for } x \text{ in } h(x) = \sqrt{x+4} \\ &= \sqrt{2x+7} \end{aligned}$$

4. To find $(f \circ f)(x)$, we substitute the function f into itself.

$$\begin{aligned} (f \circ f)(x) &= f(f(x)) && \text{by definition} \\ &= f(2x+3) && \text{since } f(x) = 2x+3 \\ &= 2(2x+3)+3 && \text{substitute } (2x+3) \text{ for } x \text{ in } f(x) = 2x+3 \\ &= 4x+9 \end{aligned}$$

5. Our next function, $(g \circ g)(x)$, is defined as $g(g(x))$ from which we get

$$
\begin{aligned}
g(g(x)) &= g\left(\frac{1}{x-1}\right) && \text{since } g(x)=\frac{1}{x-1}\\
&= \frac{1}{\left(\frac{1}{x-1}\right)-1} && \text{substitute}\left(\frac{1}{x-1}\right)\text{for } x \text{ in } g(x)=\frac{1}{x-1}\\
&= \frac{1}{\left(\frac{1}{x-1}-1\right)}\cdot\frac{(x-1)}{(x-1)} && \text{simplify complex fraction}\\
&= \frac{x-1}{1-(x-1)}\\
&= \frac{x-1}{2-x}
\end{aligned}
$$

6. We find an expression for $(h\circ g\circ f)(x)$ by rewriting the function as $(h\circ(g\circ f))(x)$[22] to get

$$
\begin{aligned}
(h\circ g\circ f)(x) &= (h\circ(g\circ f))(x)\\
&= h((g\circ f)(x)) && \text{by definition}\\
&= h\left(\frac{1}{2x+2}\right) && \text{from part 1}\\
&= \sqrt{\left(\frac{1}{2x+2}\right)+4} && \text{substitute}\left(\frac{1}{2x+2}\right)\text{for } x \text{ in } h(x)=\sqrt{x+4}\\
&= \sqrt{\frac{1}{2x+2}+\frac{4\cdot(2x+2)}{2x+2}} && \text{obtain common denominator}\\
&= \sqrt{\frac{8x+9}{2x+2}}
\end{aligned}
$$

□

In the previous example, we found that $(g\circ f)(x)=\frac{1}{2x+2}$ while $(f\circ g)(x)=\frac{3x-1}{x-1}$. In general, when we compose two functions, the order matters. Another observation from this example is the composition of a function with itself. Composing a function with itself is called 'iterating' the function.

Domains of Composite Functions

We continue with composite functions and their domains. From **Definifion 1.11**, we write the composite function $f\circ g$, with an input x, as $f(g(x))$, and we can see right away that x must be in the domain of g in order to evaluate the inside function. We can also see that $g(x)$ must be in the domain of f; otherwise the evaluation of the outside function cannot be completed. Thus, the domain of $f\circ g$ consists

---

[22] This is equivalent to $((h\circ g)\circ f)(x)$. Try it!

of only those inputs in the domain of g that produce outputs belonging to the domain of f. We can say that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Example 1.4.7. Find the domain of $(f \circ g)(x)$ where $f(x)=\dfrac{5}{x-1}$ and $g(x)=\dfrac{4}{3x-2}$.

Solution. To determine the domain of $(f \circ g)(x)=f(g(x))$, we begin with the domain of the inside function, $g(x)=\dfrac{4}{3x-2}$. The domain of $g(x)$ consists of all real numbers except $x=\dfrac{2}{3}$, since that input value would cause us to divide by zero.

By writing the outer function as $f(g(x))=\dfrac{5}{g(x)-1}$, we can see that we also need to exclude from the domain any values of x for which $g(x)=1$:

$$\begin{aligned} g(x)&=1 \\ \frac{4}{3x-2}&=1 \\ 4&=3x-2 \\ 6&=3x \\ x&=2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $x=\dfrac{2}{3}$ and $x=2$. We can write this in interval notation as $\left(-\infty,\dfrac{2}{3}\right)\cup\left(\dfrac{2}{3},2\right)\cup(2,\infty)$.

□

Example 1.4.8. Find the domain of $(g \circ f)(x)$ where $f(x)=\sqrt{3-x}$ and $g(x)=\sqrt{x+2}$.

Solution. In this example, since $(g \circ f)(x)=g(f(x))$, the inside function is $f(x)$. To determine the domain of f, since we cannot take the square root of a negative number, we look for values of x for which $3-x \geq 0$.

$$\begin{aligned} 3-x&\geq 0 \\ 3&\geq x \\ x&\leq 3 \end{aligned}$$

So the domain of f includes all real numbers less than or equal to 3.

Next, we find values of x for which $f(x)$ is in the domain of g. Since $g(f(x)) = \sqrt{f(x)+2}$, we must have

$$\begin{aligned} f(x)+2 &\geq 0 \\ \left(\sqrt{3-x}\right)+2 &\geq 0 \\ \sqrt{3-x} &\geq -2 \end{aligned}$$

Knowing that a (real-valued) square root is always nonnegative, $\sqrt{3-x} \geq -2$ for all x-values for which $\sqrt{3-x}$ is defined, and so our only restriction on the domain of $g \circ f$ comes from the domain of f. This means the domain of $g \circ f$ is the same as the domain of f, which is $(-\infty, 3]$.

□

While our choice of examples for domains of composite functions is somewhat limited until we solve quadratic and rational inequalities in **Chapters 2** and **3**, the previous two examples should provide a basic understanding of the procedure for determining domains of composite functions. We move on to determining values of composite functions.

Evaluating Composite Functions

Example 1.4.9. Let $f(x) = x^2 - 4x$ and $g(x) = 2 - \sqrt{x+3}$. Find the indicated function value.

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

Solution.

1. Using **Definition 1.11**, $(g \circ f)(1) = g(f(1))$. We find $f(1) = -3$, so

$$\begin{aligned} (g \circ f)(1) &= g(f(1)) \\ &= g(-3) \\ &= 2 - \sqrt{(-3)+3} \\ &= 2 \end{aligned}$$

2. As before, we use **Definition 1.11** to write $(f \circ g)(1) = f(g(1))$. We find $g(1) = 0$, so

$$\begin{aligned} (f \circ g)(1) &= f(g(1)) \\ &= f(0) \\ &= (0)^2 - 4(0) \\ &= 0 \end{aligned}$$

3. Once more, **Definition 1.11** tells us $(g \circ g)(6) = g(g(6))$. That is, we evaluate g at 6, then plug that result back into g. Since $g(6) = -1$,

$$\begin{aligned}(g \circ g)(6) &= g(g(6)) \\ &= g(-1) \\ &= 2 - \sqrt{(-1)+3} \\ &= 2 - \sqrt{2}\end{aligned}$$

□

In the previous example, an alternate method would be to first determine the compositions $(g \circ f)(x)$, $(f \circ g)(x)$, and $(g \circ g)(x)$, and then evaluate the resulting expressions at 1, 1, and 6, respectively. We may also find values of function operations where the function values are defined in a table, as follows.

Example 1.4.10. Use the following table to compute the function values.

x	-2	-1	0	1	2
$f(x)$	1	0	-4	2	5
$g(x)$	3	2	7	0	-2

1. $(f \cdot g)(-1)$
2. $(f \circ g)(-1)$
3. $(g \circ g)(2)$

Solution.

1. Since $(f \cdot g)(-1)$ is the product $f(-1) \cdot g(-1)$, we look in the table for the column containing an x-value of -1. We see that when $x = -1$, $f(x) = 0$ and $g(x) = 2$. So we have

$$\begin{aligned}(f \cdot g)(-1) &= f(-1) \cdot g(-1) \\ &= (0)(2) \\ &= 0\end{aligned}$$

2. Next we compute $(f \circ g)(-1)$. This expression is similar to $(f \cdot g)(-1)$, but the open circle tells us that we have composition of functions, not multiplication. We find

$$\begin{aligned}(f \circ g)(-1) &= f(g(-1)) \\ &= f(2)\end{aligned}$$

From the table, $f(2) = 5$, so we have $(f \circ g)(-1) = 5$.

3. To compute $(g \circ g)(2)$, we find

$$\begin{aligned}(g \circ g)(2) &= g(g(2)) \\ &= g(-2) \\ &= 3\end{aligned}$$

□

Function composition is often used to relate two quantities that may not be directly related, but have a variable in common, as illustrated in our next example.

Example 1.4.11. The surface area S of a sphere is a function of its radius r, and is given by the formula $S(r)=4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t)=3t^2$, where t is measured in seconds, $t\geq 0$, and r is measured in inches. Find and interpret $(S\circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$, and feed the radius, r, into $S(r)$ to find the surface area at time t. From this we see that the surface area S is ultimately a function of time t and we find

$$\begin{aligned}(S\circ r)(t)&=S(r(t))\\&=4\pi(r(t))^2\\&=4\pi(3t^2)^2\\&=36\pi t^4\end{aligned}$$

The formula $(S\circ r)(t)=36\pi t^4$ allows us to compute the surface area directly, given the time, without going through the 'middle man' r.

□

Decomposing a Composite Function

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions as our last example illustrates.

Example 1.4.12. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x)=|3x-1|$
2. $G(x)=\dfrac{2}{x^2+1}$

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function $F(x)=|3x-1|$ as $F=g\circ f$ for functions f and g. From **Definition 1.11**, we know $F(x)=g(f(x))$, where $f(x)$ is the inside function and $g(x)$ is the outside function. Looking at $F(x)=|3x-1|$ from an inside versus outside perspective, we can think

of $3x-1$ as being inside the absolute value symbols. Taking this cue, we define $f(x)=3x-1$. At this point, we have $F(x)=|f(x)|$. What is the outside function? The function that takes the absolute value of its input, $g(x)=|x|$. We check our answers for f and g.

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=|f(x)|\\&=|3x-1|\\&=F(x)\end{aligned}$$

This verifies the solution $F=g\circ f$ where $f(x)=3x-1$ and $g(x)=|x|$. Note that there is more than one solution here. For example, $f(x)=3x$ and $g(x)=|x-1|$ would also give us the desired result.

2. We attack deconstructing $G(x)=\dfrac{2}{x^2+1}$ from an operational approach. Given an input x, the first step is to square x, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of three functions: f, g, and h. Our first function, f, is the function that squares its input, $f(x)=x^2$. The next function is the function that adds 1 to its input, $g(x)=x+1$. Our last function takes its input and divides it into 2, $h(x)=\dfrac{2}{x}$. The claim is that $G=h\circ g\circ f$. We test the claim as follows.

$$\begin{aligned}(h\circ g\circ f)(x)&=h(g(f(x)))\\&=h(g(x^2))\\&=h(x^2+1)\\&=\frac{2}{x^2+1}\\&=G(x)\end{aligned}$$

We have verified our solution of $G=h\circ g\circ f$ for $f(x)=x^2$, $g(x)=x+1$ and $h(x)=\dfrac{2}{x}$.

□

Note: We could have decomposed $G(x)=\dfrac{2}{x^2+1}$ using only two functions. Let $k(x)=x^2+1$. Then

$$\begin{aligned}(h \circ k)(x) &= h(k(x)) \\ &= h\left(x^2+1\right) \\ &= \frac{2}{x^2+1} \\ &= G(x)\end{aligned}$$

1.4 Exercises

1. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

2. How do you find the domain for the composition of two functions, $f \circ g$?

In Exercises 3 – 8, use the given function f to find and simplify the following:

(a) $f(4x)$ (b) $4f(x)$ (c) $f(-x)$

(d) $f(x-4)$ (e) $f(x)-4$ (f) $f(x^2)$

3. $f(x)=2x+1$

4. $f(x)=3-4x$

5. $f(x)=2-x^2$

6. $f(x)=x^2-3x+2$

7. $f(x)=\dfrac{x}{x-1}$

8. $f(x)=\dfrac{2}{x^3}$

In Exercises 9 – 16, use the given function f to find and simplify the following:

(a) $f(2a)$ (b) $2f(a)$ (c) $f\left(\dfrac{2}{a}\right)$

(d) $\dfrac{f(a)}{2}$ (e) $f(a+h)$ (f) $f(a)+f(h)$

9. $f(x)=2x-5$

10. $f(x)=5-2x$

11. $f(x)=2x^2-1$

12. $f(x)=3x^2+3x-2$

13. $f(x)=\sqrt{2x+1}$

14. $f(x)=117$

15. $f(x)=\dfrac{x}{2}$

16. $f(x)=\dfrac{2}{x}$

In Exercises 17 – 26, use the pair of functions f and g to find the following values, if they exist.

(a) $(f+g)(2)$ (b) $(f-g)(-1)$ (c) $(g-f)(1)$

(d) $(f \cdot g)\left(\dfrac{1}{2}\right)$ (e) $\left(\dfrac{f}{g}\right)(0)$ (f) $\left(\dfrac{g}{f}\right)(-2)$

17. $f(x)=3x+1,\ g(x)=4-x$

18. $f(x)=x^2,\ g(x)=-2x+1$

19. $f(x)=x^2-x,\ g(x)=12-x^2$

20. $f(x)=2x^3,\ g(x)=-x^2-2x-3$

21. $f(x)=\sqrt{x+3},\ g(x)=2x-1$

22. $f(x)=\sqrt{4-x},\ g(x)=\sqrt{x+2}$

23. $f(x)=2x,\ g(x)=\dfrac{1}{2x+1}$

24. $f(x)=x^2,\ g(x)=\dfrac{3}{2x-3}$

25. $f(x)=x^2,\ g(x)=\dfrac{1}{x^2}$

26. $f(x)=x^2+1,\ g(x)=\dfrac{1}{x^2+1}$

In Exercises 27 – 36, use the pair of functions f and g to find and simplify an expression for the indicated function. Determine the domain.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f\cdot g)(x)$ (d) $\left(\dfrac{f}{g}\right)(x)$

27. $f(x)=2x+1,\ g(x)=x-2$

28. $f(x)=1-4x,\ g(x)=2x-1$

29. $f(x)=x^2,\ g(x)=3x-1$

30. $f(x)=x^2-x,\ g(x)=7x$

31. $f(x)=x^2-4,\ g(x)=3x+6$

32. $f(x)=-x^2+x+6,\ g(x)=x^2-9$

33. $f(x)=\dfrac{x}{2},\ g(x)=\dfrac{2}{x}$

34. $f(x)=x-1,\ g(x)=\dfrac{1}{x-1}$

35. $f(x)=x,\ g(x)=\sqrt{x+1}$

36. $f(x)=\sqrt{x-5},\ g(x)=f(x)=\sqrt{x-5}$

In Exercises 37 – 54, find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the given function.

37. $f(x)=2x-5$

38. $f(x)=-3x+5$

39. $f(x)=6$

40. $f(x)=3x^2-x$

41. $f(x)=-x^2+2x-1$

42. $f(x)=4x^2$

43. $f(x)=x-x^2$

44. $f(x)=x^3+1$

45. $f(x)=mx+b$ where $m\neq 0$

46. $f(x)=ax^2+bx+c$ where $a\neq 0$

47. $f(x)=\dfrac{2}{x}$

48. $f(x)=\dfrac{3}{1-x}$

49. $f(x) = \dfrac{1}{x^2}$

50. $f(x) = \dfrac{2}{x+5}$

51. $f(x) = \dfrac{1}{4x-3}$

52. $f(x) = \dfrac{3x}{x+1}$

53. $f(x) = \dfrac{x}{x-9}$

54. $f(x) = \dfrac{x^2}{2x+1}$

In Exercises 55 – 69, use the following table of function values to compute the indicated value, if it exists.

x	-3	-2	-1	0	1	2	3
$f(x)$	4	2	0	1	3	4	-1
$g(x)$	-2	0	-4	0	-3	1	2

55. $(f+g)(-3)$

56. $(f \cdot g)(-1)$

57. $(g-f)(3)$

58. $\left(\dfrac{f}{g}\right)(-1)$

59. $\left(\dfrac{f}{g}\right)(2)$

60. $\left(\dfrac{g}{f}\right)(-1)$

61. $(f \circ g)(3)$

62. $f(g(-1))$

63. $(f \circ f)(0)$

64. $g(f(-3))$

65. $(g \circ g)(-2)$

66. $(g \circ f)(-2)$

67. $f(f(f(-1)))$

68. $f(f(f(f(f(1)))))$

69. $(\underbrace{g \circ g \circ \cdots \circ g}_{\text{n times}})(0)$

In Exercises 70 – 81, use the given pair of functions to find the following values, if they exist.

(a) $(g \circ f)(0)$

(b) $(f \circ g)(-1)$

(c) $(f \circ f)(2)$

(d) $(g \circ f)(-3)$

(e) $(f \circ g)\left(\dfrac{1}{2}\right)$

(f) $(f \circ f)(-2)$

70. $f(x) = x^2$, $g(x) = 2x+1$

71. $f(x) = 4-x$, $g(x) = 1-x^2$

72. $f(x) = 4-3x$, $g(x) = |x|$

73. $f(x) = |x-1|$, $g(x) = x^2-5$

74. $f(x) = 4x+5$, $g(x) = \sqrt{x}$

75. $f(x) = \sqrt{3-x}$, $g(x) = x^2+1$

76. $f(x) = 6-x-x^2$, $g(x) = x\sqrt{x+10}$

77. $f(x) = \sqrt[3]{x+1}$, $g(x) = 4x^2-x$

78. $f(x) = \dfrac{3}{1-x}$, $g(x) = \dfrac{4x}{x^2+1}$

79. $f(x) = \dfrac{x}{x+5}$, $g(x) = \dfrac{2}{7-x^2}$

80. $f(x)=\dfrac{2x}{5-x^2}$, $g(x)=\sqrt{4x+1}$

81. $f(x)=\sqrt{2x+5}$, $g(x)=\dfrac{10x}{x^2+1}$

In Exercises 82 – 91, use the given pair of functions to find and simplify expressions for the following functions. State the domain of each using interval notation.

(a) $(g\circ f)(x)$ (b) $(f\circ g)(x)$ (c) $(f\circ f)(x)$

82. $f(x)=2x+3$, $g(x)=x^2-9$

83. $f(x)=x^2-x+1$, $g(x)=3x-5$

84. $f(x)=x^2-4$, $g(x)=|x|$

85. $f(x)=3x-5$, $g(x)=\sqrt{x}$

86. $f(x)=|x+1|$, $g(x)=\sqrt{x}$

87. $f(x)=|x|$, $g(x)=\sqrt{4-x}$

88. $f(x)=\dfrac{1}{x}$, $g(x)=x-3$

89. $f(x)=3x-1$, $g(x)=\dfrac{1}{x+3}$

90. $f(x)=\dfrac{2}{x}$, $g(x)=\dfrac{3}{x+5}$

91. $f(x)=\dfrac{x}{2x+1}$, $g(x)=\dfrac{2x+1}{x}$

92. Use the functions $f(x)=\dfrac{1}{x^2-8}$ and $g(x)=\sqrt{x+2}$ to find and simplify $(f\circ g)(x)$.

In Exercises 93 – 96, use the given pair of functions to find and simplify expressions for the following composite functions. You are not required to find the domain.

(a) $(g\circ f)(x)$ (b) $(f\circ g)(x)$ (c) $(f\circ f)(x)$

93. $f(x)=3-x^2$, $g(x)=\sqrt{x+1}$

94. $f(x)=x^2-x-1$, $g(x)=\sqrt{x-5}$

95. $f(x)=\dfrac{2x}{x^2-4}$, $g(x)=\sqrt{1-x}$

96. $f(x)=\sqrt{2-4x}$, $g(x)=-\dfrac{3}{x}$

In Exercises 97 – 102, use $f(x)=-2x$, $g(x)=\sqrt{x}$, and $h(x)=|x|$ to find and simplify expressions for the following composite functions. State the domain of each using interval notation.

97. $(h\circ g\circ f)(x)$ 98. $(h\circ f\circ g)(x)$ 99. $(g\circ f\circ h)(x)$

100. $(g\circ h\circ f)(x)$ 101. $(f\circ h\circ g)(x)$ 102. $(f\circ g\circ h)(x)$

In Exercises 103 – 114, write the given function as a composition of two or more non-identity[23] functions. (There are several correct answers so check your answer using function composition.)

103. $p(x)=(2x+3)^3$ 104. $P(x)=(x^2-x+1)^5$ 105. $h(x)=\sqrt{2x-1}$

106. $H(x)=|7-3x|$ 107. $r(x)=\dfrac{2}{5x+1}$ 108. $R(x)=\dfrac{7}{x^2-1}$

109. $q(x)=\dfrac{|x|+1}{|x|-1}$ 110. $Q(x)=\dfrac{2x^3+1}{x^3-1}$ 111. $v(x)=\dfrac{2x+1}{3-4x}$

112. $V(x)=\dfrac{x^2}{x^4+1}$ 113. $w(x)=\dfrac{1}{(x-2)^3}$ 114. $W(x)=\left(\dfrac{1}{2x-3}\right)^2$

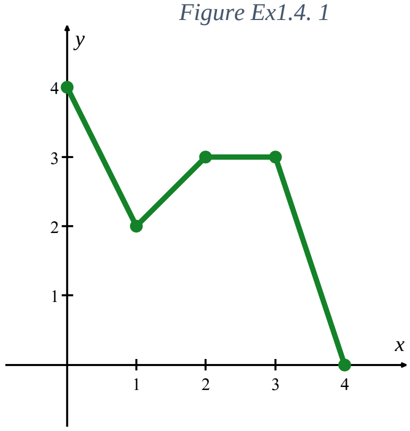
115. Write the function $F(x)=\sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

116. Let $g(x)=-x$, $h(x)=x+2$, $j(x)=3x$, and $k(x)=x-4$. In what order must these functions be composed with $f(x)=\sqrt{x}$ to create $F(x)=3\sqrt{-x+2}-4$?

117. What linear functions could be used to transform $f(x)=x^3$ into $F(x)=-\frac{1}{2}(2x-7)^3+1$? What is the proper order of composition?

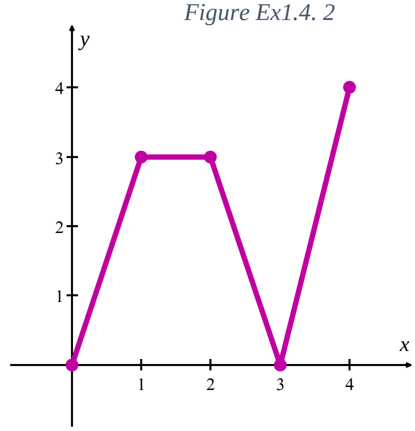
In Exercises 118 – 123, use the graphs of $y=f(x)$ and $y=g(x)$ below to find the function value.

Figure Ex1.4. 1



$y=f(x)$

Figure Ex1.4. 2



$y=g(x)$

118. $(g\circ f)(1)$ 119. $(f\circ g)(3)$ 120. $(g\circ f)(2)$

[23] The identity function is $I(x)=x$.

121. $(f \circ g)(0)$ 122. $(f \circ f)(1)$ 123. $(g \circ g)(1)$

124. The volume V of a cube is a function of its side length x. Let's assume that $x = t + 1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

125. A store offers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Use function composition to find a price function $P(x)$ that computes the final price of the item in terms of the original price x.

126. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

127. Use the function you found in the previous exercise to find the area of the ripple after 5 minutes.

128. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours. Find the composite function $N(T(t))$, and use it to find the time when the bacteria count reaches 6752.

129. Discuss with your classmates how real-world processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

1.5 Inverses of Functions

Learning Objectives

- Verify that two functions are inverses of each other.
- Determine if a function is one-to-one.
- Use the graph of a one-to-one function to graph its inverse function.
- Find the inverse of a one-to-one function.

Thinking of a function as a process, we seek another function that might reverse the process. As in real life, we will find that some processes (like the process of putting socks on followed by putting shoes on) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function that is reversible, $f(x)=3x+4$. Thinking of f as a process, we start with an input x and apply two steps:

1. Multiply by 3.
2. Add 4.

To reverse this process, we seek a function g that will undo each of these steps by taking the output $3x+4$ from f and returning the input x. If we think of the real-world reversible two-step process of first putting on socks and then putting on shoes, to reverse the process we first take off the shoes and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, g should

1. Subtract 4.
2. Divide by 3.

Following this procedure, we get $g(x)=\frac{x-4}{3}$. Let's check to see if the function g does the job. For the specific value of $x=5$, we have $f(5)=3(5)+4=19$. We next find $g(19)=\frac{19-4}{3}=5$, which is the original input to f. So g works for an x-value of 5. Now, to verify that g is the inverse of f, we must show that it works for any value of x.

Note that, for any real number x, we have

$$\begin{aligned} g\left(f(x)\right) &= g(3x+4) \\ &= \frac{(3x+4)-4}{3} \\ &= \frac{3x}{3} \\ &= x \end{aligned}$$

This is the original input to f. So g is indeed the inverse of f. By carefully examining $g\left(f(x)\right)$, we see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x)=\frac{x-4}{3}$, and put it into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right)+4 = (x-4)+4 = x$.

Defining Inverse Functions

Using the language of function composition developed in **Section 1.4**, the statements $g\left(f(x)\right)=x$ and $f(g(x))=x$ can be written as $(g\circ f)(x)=x$ and $(f\circ g)(x)=x$, respectively. The main idea is that g takes the outputs from f and returns them to their respective inputs and, conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of this section.

Definition 1.12. A function g is the **inverse** of a function f if

1. $(g\circ f)(x)=x$ for all x in the domain of f **and**
2. $(f\circ g)(x)=x$ for all x in the domain of g.

If such a function g exists, function f is said to be **invertible** and function f is also the inverse of function g; that is, f and g are **inverses** of each other.

Example 1.5.1. Verify that functions $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x}-2$ are inverses of each other.

Solution. We first show that $(g\circ f)(x)=x$ for all x in the domain of f, which is $\{x \mid x\neq -2\}$. For $x\neq -2$,

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g\left(\frac{1}{x+2}\right)\\&=\frac{1}{\left(\frac{1}{x+2}\right)}-2\\&=x+2-2\\&=x\end{aligned}$$

With $(g\circ f)(x)=x$, we have met the first condition of **Definition 1.12**. Now, we show that $(f\circ g)(x)=x$ for all x in the domain of g, which is $\{x\,|\,x\neq 0\}$. For $x\neq 0$,

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f\left(\frac{1}{x}-2\right)\\&=\frac{1}{\left(\frac{1}{x}-2\right)+2}\\&=\frac{1}{\left(\frac{1}{x}\right)}\\&=x\end{aligned}$$

With $(f\circ g)(x)=x$, we have met the second condition of **Definition 1.12**. So, we have verified that f and g are inverses of each other.

□

At this point, we may wonder if every function is invertible or if there is a function which, like cooking a steak, is not reversible.

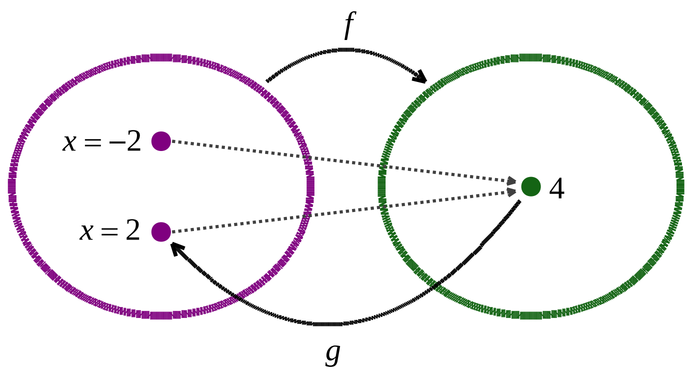
Functions that are Invertible

Let us consider the function $f(x)=x^2$ with domain $(-\infty,\infty)$. Is there a function that is the inverse of the function f? A likely candidate for reversing the process of squaring is $g(x)=\sqrt{x}$. By **Definition 1.12**, $(g\circ f)(x)$ must be equal to x for all x in the domain of f. We find that this is not always the case. For example, when $x=-2$,

$$\begin{aligned} g\left(f(-2)\right) &= g\left((-2)^2\right) \\ &= g(4) \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

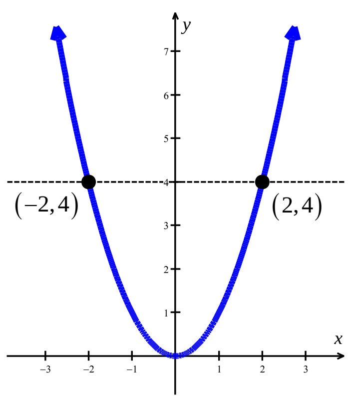
Since $g\left(f(-2)\right) \neq -2$, function g is not the inverse of f. For further insight into the situation, we note that $f(2)=4$, from which it follows that $g\left(f(2)\right)=2$. This is presented schematically in the following picture.

Figure 1.5. 1



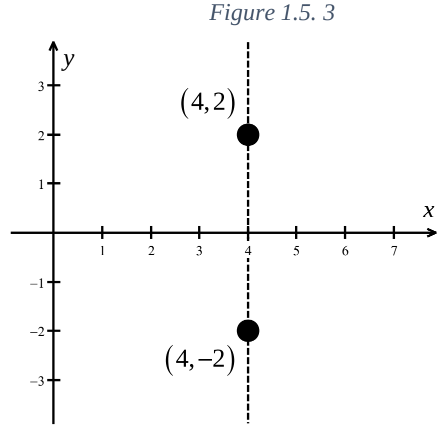
We see from the diagram that both $f(-2)$ and $f(2)$ are 4. By definition, a function can take 4 back to only −2 or 2, not both. From a graphical standpoint, points $(-2,4)$ and $(2,4)$, which lie on a horizontal line, are on the graph of f, but by the vertical line test, no function can contain both points $(4,-2)$ and $(4,2)$.

Figure 1.5. 2



$y = f(x) = x^2$

Figure 1.5. 3



A function?

Therefore, for a function to have an inverse, different inputs must go to different outputs or else we will run into the same problem we did with $f(x)=x^2$. We give this property a name.

Definition 1.13. A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Note that a function is not one-to-one if there exist c and d in its domain such that $c \neq d$ and $f(c) = f(d)$. Graphically, we detect one-to-one functions using the test below.

Theorem 1.2. The Horizontal Line Test: A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.[24]

A function being one-to-one is enough to guarantee invertibility since every output value corresponds to a unique input value and, hence, the process is reversible. We summarize these results below.

Theorem 1.3. Equivalent Conditions for Invertibility: Suppose f is a function. The following statements are equivalent.

- f is invertible.
- f is one-to-one.
- The graph of f passes the horizontal line test.

We put this result to work in the next example.

Example 1.5.2. Determine if the following functions are one-to-one in two ways: (a) analytically using **Definition 1.13** and (b) graphically using the horizontal line test.

1. $f(x) = \dfrac{1}{x+2}$
2. $g(x) = x^2 - 2x + 4$

Solution.

1. (a) We begin with the assumption that $f(c) = f(d)$ and try to show $c = d$.

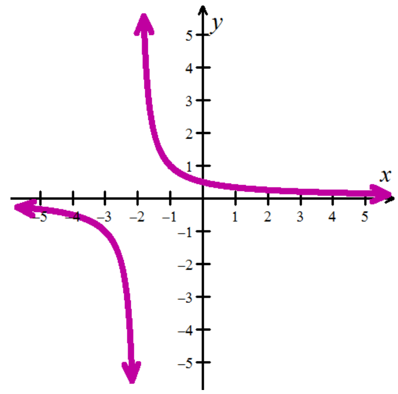
$$\begin{aligned} f(c) &= f(d) \\ \frac{1}{c+2} &= \frac{1}{d+2} \qquad \text{since } f(x) = \frac{1}{x+2} \\ (1)(d+2) &= (1)(c+2) \\ d+2 &= c+2 \\ d &= c \end{aligned}$$

We have shown that f is one-to-one.

[24] This is true in the Cartesian coordinate system, but not in the polar coordinate system, which is discussed in Trigonometry and Calculus.

(b) We can graph $f(x)=\dfrac{1}{x+2}$ through transformations of the toolkit function $y=\dfrac{1}{x}$. To graph $y=f(x)$, we shift $y=\dfrac{1}{x}$ to the left by 2 units. We see that the graph of f passes the horizontal line test, verifying f is one-to-one.

Figure 1.5. 4



$y=f(x)$

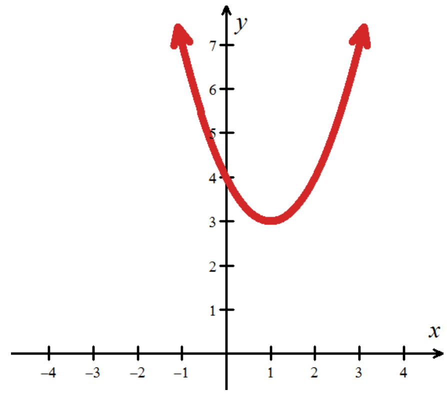
2. (a) As mentioned earlier, a function f is not one-to-one if there exist numbers c and d in its domain such that $c \neq d$ and $f(c)=f(d)$. For $g(x)=x^2-2x+4$, since two different input values, $x=0$ and $x=2$, result in the same output value, $g(0)=0^2-2\cdot 0+4=4$ and $g(2)=2^2-2\cdot 2+4=4$, g is not a one-to-one function. Although we have solved this problem, we will show a method to discover all such input values by starting with $g(c)=g(d)$. As we work our way through this equation, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a zero on the right, and factor accordingly.

$$\begin{aligned}
g(c)&=g(d)\\
c^2-2c+4&=d^2-2d+4 && \text{since } g(x)=x^2-2x+4\\
c^2-2c&=d^2-2d && \text{subtract 4}\\
c^2-d^2-2c+2d&=0\\
(c+d)(c-d)-2(c-d)&=0\\
((c+d)-2)(c-d)&=0 && \text{factor by grouping}
\end{aligned}$$

So, $g(c)=g(d)$ when $((c+d)-2)=0$ or $c-d=0$. That is, when $c=2-d$ or $c=d$. From setting d equal to zero, we get the above input values of $c=2$ or $c=0$. We may find additional pairs of input values that result in the same output value by assigning other values to d, such as $d=3$, giving us input values of $c=-1$ or $c=3$, for an output value of 7.

(b) We note that $g(x) = x^2 - 2x + 4$ can be written as $g(x) = (x-1)^2 + 3$ and as such may be graphed as a transformation of the toolkit function $y = x^2$. We see immediately from the graph[25] that g is not one-to-one since there are horizontal lines that intersect the graph more than once.

Figure 1.5. 5



$y = g(x)$

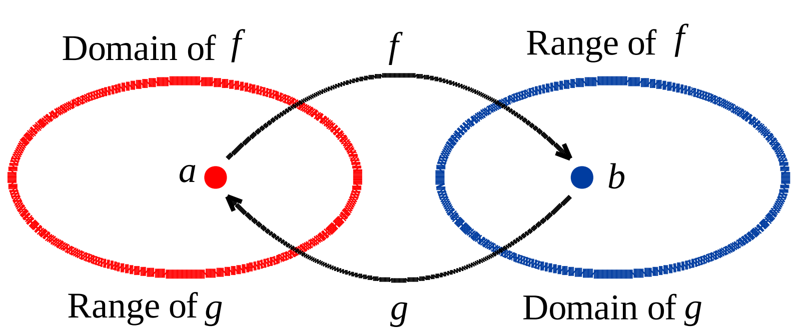
□

Looking back at the toolkit functions in **Section 1.2**, the horizontal line test will verify that the identity function, the cubing function, the reciprocal function, the square root function, and the cube root function are one-to-one. Furthermore, the six transformations discussed in **Section 1.3** will preserve that one-to-one property. Take some time to verify this on your own. We next observe some properties of inverse functions that follow from **Definition 1.12**.

Properties of Inverse Functions

In writing $(g \circ f)(x) = g(f(x)) = x$, for all x in the domain of f, it is implied that every input of f is an output of g, while every output of f is an input of g. Likewise, $(f \circ g)(x) = f(g(x)) = x$, for all x in the domain of g, implies that every input of g is an output of f, while every output of g is an input of f. Taken together, it means that the domain of f is the range of g and the range of f is the domain of g. We can visualize the situation in the following diagram.

Figure 1.5. 6



[25] Practice with graphing quadratic functions is coming in **Chapter 2**.

We next formalize this concept that inverse functions exchange inputs and outputs.

Theorem 1.4. Properties of Inverse Functions: Suppose functions f and g are inverses of each other.

- The range[26] of f is the domain of g and the domain of f is the range of g.
- $f(a)=b$ if and only if $g(b)=a$.
- The point (a,b) is on the graph of f if and only if the point (b,a) is on the graph of g.

One implication of **Theorem 1.4** is that an inverse function g is uniquely defined by the invertible function f, since we know the domain of g and all its function values. Notice that points (a,b) and (b,a) are symmetric about the line $y=x$ since this line is a perpendicular bisector[27] of the line segment connecting them, as shown on the following graph, on the left.

Figure 1.5. 7

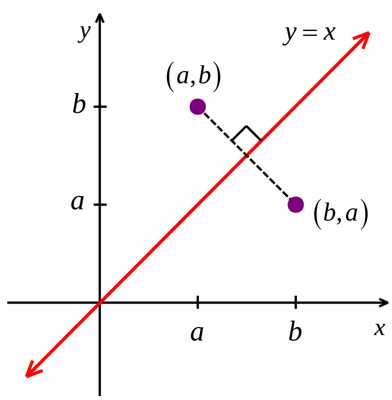
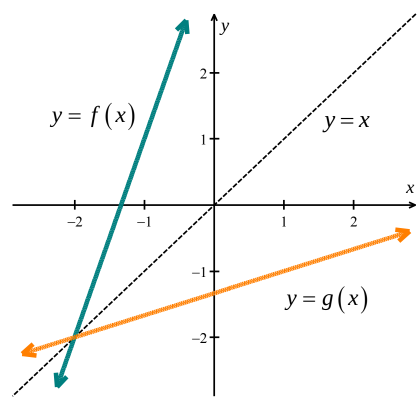


Figure 1.5. 8



Thus, another implication of **Theorem 1.4** is that graphs of an invertible function and its inverse are symmetric about the line $y=x$. We demonstrate this property above, to the right, with graphs of the functions $f(x)=3x+4$ and $g(x)=\dfrac{x-4}{3}$ that are inverses of each other. We present the following theorem to summarize the uniqueness and graphing properties of inverse functions.

[26] Recall that this is the set of all outputs of a function.

[27] This can be verified algebraically by showing that the slope of the line segment is -1 and the distance from (a,b) to a point on the line is the same as the distance from (b,a) to that same point on the line.

Theorem 1.5. Uniqueness of Inverse Functions and Their Graphs: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f inverse').
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

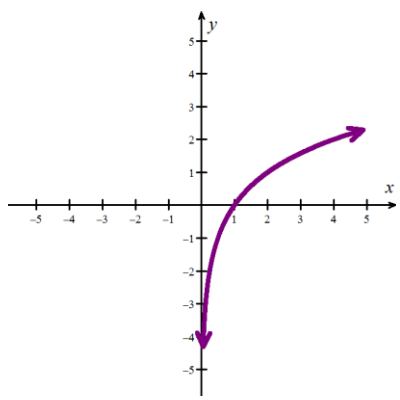
For a function, the notation f^{-1} is not $\frac{1}{f}$. For example, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Finding the Inverse of a One-to-One Function

We begin by using the graph of a function to sketch the inverse function. From **Theorem 1.5**, the graph of the inverse function, f^{-1}, is the reflection of the graph of f across the line $y = x$.

Example 1.5.3. Given the following graph of f, sketch a graph of f^{-1}.

Figure 1.5. 9

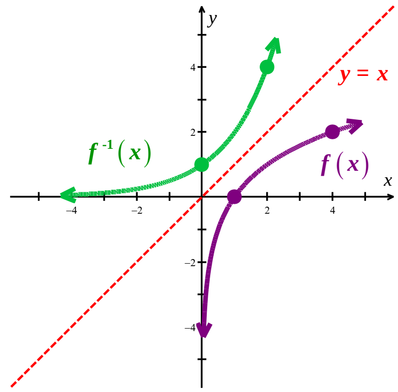


$y = f(x)$

Solution. We first observe that f is a one-to-one function, so will have an inverse. We also note that the graph has an apparent domain of $(0,\infty)$ and range of $(-\infty,\infty)$. From **Theorem 1.4**, the inverse will have a domain of $(-\infty,\infty)$ and range of $(0,\infty)$.

It is easy to see that the points $(1,0)$ and $(4,2)$ are on the graph of f. If we reflect the graph of $y = f(x)$ across the line $y = x$, the point $(1,0)$ reflects to $(0,1)$ and the point $(4,2)$ reflects to $(2,4)$. We next sketch the inverse along with the original function.

Figure 1.5. 10



□

To find the inverse of a function defined by an equation, we follow a general methodology, as outlined below. **Theorem 1.4** tells us the equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of the following method for determining the inverse.

Steps for Finding the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Interchange x and y.
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$.

Note that we could have simply written 'Solve $x = f(y)$ for y' and be done with it. The act of interchanging the x and y is there to remind us that we are finding the inverse function by switching the inputs and the outputs.

Example 1.5.4. Find the inverse of the one-to-one function $f(x) = \dfrac{2-3x}{4}$ and graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

Solution.

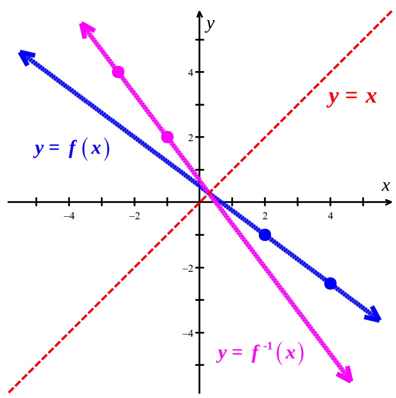
We will use the steps, outlined above, to find the inverse of $f(x) = \dfrac{2-3x}{4}$.

$$\begin{aligned} y &= \frac{2-3x}{4} && \text{from } y = f(x) \\ x &= \frac{2-3y}{4} && \text{interchange } x \text{ and } y \\ 4x &= 2-3y \\ 4x-2 &= -3y \\ \frac{4x-2}{-3} &= y \end{aligned}$$

We can rewrite the resulting equation as $y=-\frac{4}{3}x+\frac{2}{3}$ and, last of all, will refer to this inverse function as $f^{-1}(x)=-\frac{4}{3}x+\frac{2}{3}$.

Using points $(2,-1)$ and $\left(4,-\frac{5}{2}\right)$ on the graph of f, along with corresponding points $(-1,2)$ and $\left(-\frac{5}{2},4\right)$ on the graph of f^{-1}, the two functions are graphed below. Notice that their graphs are reflections of each other across the line $y=x$.

Figure 1.5. 11



□

Example 1.5.5. Find the inverse of the one-to-one function $g(x)=\frac{2x}{1-x}$ and use it to determine the range of the function g. Verify your answer analytically.

Solution. We find the inverse of $g(x)=\frac{2x}{1-x}$, using the steps that we have outlined above.

$$
\begin{aligned}
y &= \frac{2x}{1-x} && \text{from } y = g(x) \\
x &= \frac{2y}{1-y} && \text{interchange } x \text{ and } y \\
x(1-y) &= 2y && \text{multiply by } (1-y) \\
x - xy &= 2y && \text{expand} \\
x &= 2y + xy && \text{isolate terms containing } y \text{ on one side} \\
x &= (2+x)y && \text{factor} \\
\frac{x}{2+x} &= y && \text{solve for } y
\end{aligned}
$$

We have $g^{-1}(x) = \dfrac{x}{x+2}$. The range of g is the domain of g^{-1}, which is $(-\infty,-2)\cup(-2,\infty)$.

To check analytically that we have the correct inverse, we first verify that $(g^{-1} \circ g)(x) = x$ for all x in the domain of g; that is, for all $x \neq 1$.

$$
\begin{aligned}
(g^{-1} \circ g)(x) &= g^{-1}(g(x)) \\
&= g^{-1}\left(\frac{2x}{1-x}\right) \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} && \text{since } g^{-1}(x) = \frac{x}{x+2} \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} \cdot \frac{(1-x)}{(1-x)} && \text{clear denominators} \\
&= \frac{2x}{2x + 2(1-x)} \\
&= \frac{2x}{2} \\
&= x
\end{aligned}
$$

Next, we verify that $(g \circ g^{-1})(x) = x$ for all $x \neq 2$.

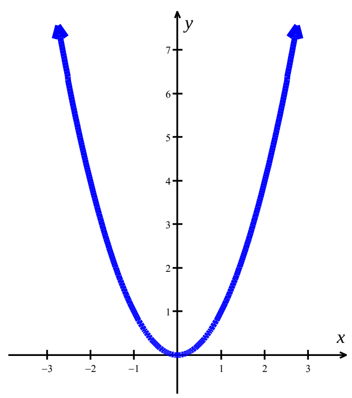
$$
\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} && \text{since } g(x)=\frac{2x}{1-x} \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} && \text{clear denominators} \\
&= \frac{2x}{(x+2)-x} \\
&= \frac{2x}{2} \\
&= x
\end{aligned}
$$

We have verified that g and g^{-1} are inverses of each other.

□

We return to $f(x)=x^2$. We know that f, with domain $(-\infty,\infty)$, is not one-to-one, and thus is not invertible. However, if we restrict the domain of f, we can produce a new function g that is one-to-one. If we define $g(x)=x^2, x\geq 0$, then we have the following.

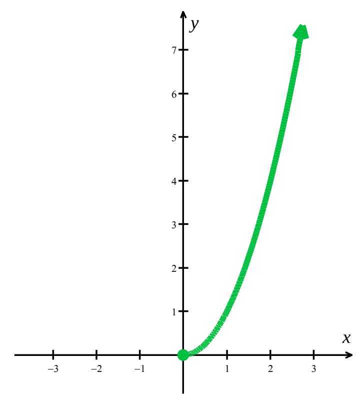
Figure 1.5. 12



$y=f(x)=x^2$

restrict domain to $x \geq 0$

Figure 1.5. 13



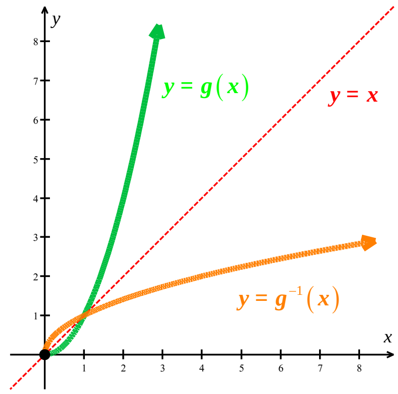
$y=g(x)=x^2,\ x\geq 0$

The graph of g passes the horizontal line test. To find an inverse of g, we proceed as usual.

$$
\begin{aligned}
y &= g(x) \\
y &= x^2,\ x \ge 0 \\
x &= y^2,\ y \ge 0 \qquad \text{interchange } x \text{ and } y \\
y &= \pm\sqrt{x},\ y \ge 0
\end{aligned}
$$

Since $y \ge 0$, we find $y = \sqrt{x}$, and so $g^{-1}(x) = \sqrt{x}$. With the restriction on the domain of g, we find that $(g^{-1} \circ g)(x) = x$ and $(g \circ g^{-1})(x) = x$. Graphing g and g^{-1} on the same set of axes shows they are reflections across the line $y = x$.

Figure 1.5. 14



Our next example continues the theme of domain restriction.

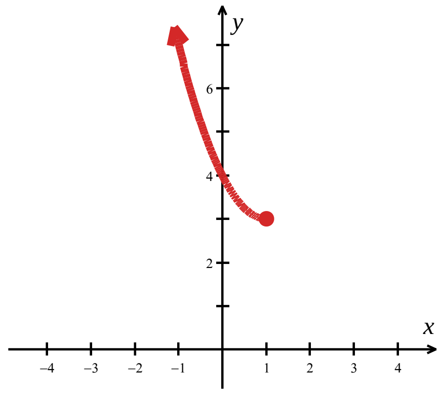
Example 1.5.6. Graph the following functions to show they are one-to-one. Find their inverses.

1. $j(x) = x^2 - 2x + 4$, $x \le 1$
2. $k(x) = \sqrt{x+2} - 1$

Solution.

1. The function j is a restriction of the function g from **Example 1.5.2**. Since the domain of j is restricted to $x \le 1$, we are selecting only the left half of the parabola. We see that the graph of j passes the horizontal line test and thus j is invertible.

Figure 1.5. 15



$y = j(x)$

We now use our algorithm to find $j^{-1}(x)$.[28]

$$y = x^2 - 2x + 4,\ x \le 1 \qquad \text{from } y = j(x)$$

$$x = y^2 - 2y + 4,\ y \le 1 \qquad \text{interchange } x \text{ and } y$$

$$0 = y^2 - 2y + 4 - x$$

$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} \qquad \text{quadratic formula, } c = 4 - x$$

$$y = \frac{2 \pm \sqrt{4x - 12}}{2}$$

$$y = \frac{2 \pm 2\sqrt{x-3}}{2}$$

$$y = 1 \pm \sqrt{x-3}$$

$$y = 1 - \sqrt{x-3} \qquad \text{since } y \le 1$$

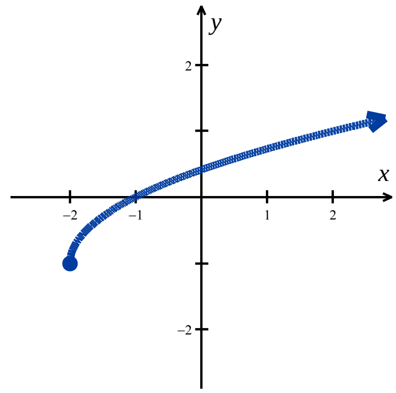
We have $j^{-1}(x) = 1 - \sqrt{x-3}$.

2. We graph $k(x) = \sqrt{x+2} - 1$ using transformations of the toolkit function $y = \sqrt{x}$.

---

[28] Here, we use the Quadratic Formula to solve for *y*. We note that you can (and should!) also consider solving for *y* by completing the square.

Figure 1.5. 16



$y = k(x)$

We next find k^{-1}.

$$
\begin{aligned}
y &= \sqrt{x+2} - 1 \quad \text{from } y = k(x) \\
x &= \sqrt{y+2} - 1 \quad \text{interchange } x \text{ and } y \\
x+1 &= \sqrt{y+2} \\
(x+1)^2 &= \left(\sqrt{y+2}\right)^2 \\
x^2 + 2x + 1 &= y + 2 \\
y &= x^2 + 2x - 1
\end{aligned}
$$

We have $k^{-1}(x) = x^2 + 2x - 1$. Noting that k^{-1} is a quadratic function, we recall seeing several quadratic functions in this section that were not one-to-one unless their domains were suitably restricted. **Theorem 1.4** tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $x \geq -1$. Our final answer for the inverse of k is $k^{-1}(x) = x^2 + 2x - 1$, $x \geq -1$. This result can be verified through showing that $(k^{-1} \circ k)(x) = x$ and $(k \circ k^{-1})(x) = x$.

□

1.5 Exercises

1. Why do we restrict the domain of the function $f(x)=x^2$ to find the function's inverse?

2. Are one-to-one functions either always increasing or always decreasing? Why or why not?

In Exercises 3 – 4, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

3. $f(x)=\sqrt[3]{x-1}$ and $g(x)=x^3+1$

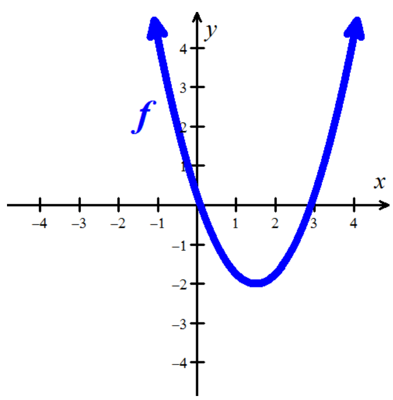
4. $f(x)=-3x+5$ and $g(x)=\dfrac{x-5}{-3}$

5. Show that the function $f(x)=3-x$ is its own inverse.

In Exercises 6 – 7, determine whether the graph represents a one-to-one function.

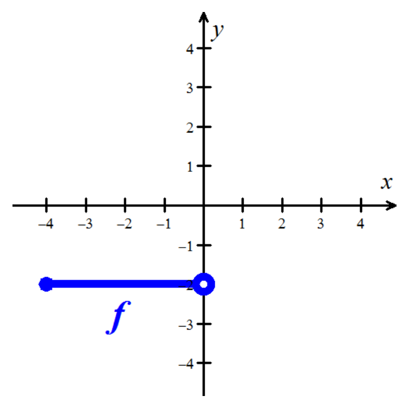
6.

Figure Ex1.5. 1



7.

Figure Ex1.5. 2



8. Determine if the table of values represents y as a function of x. If the table does represent y as a function of x, is the function one-to-one?

(a)

x	4	10	15
y	2	7	7

(b)

x	4	10	15
y	2	7	13

(c)

x	4	10	10
y	2	7	13

9. Use the following table of values for the function $f(x)$ to create a table of values for the function's inverse, $f^{-1}(x)$.

x	1	5	8	13	16
$f(x)$	2	4	9	12	15

10. Use the following table of values for the function $f(x)$ to determine the requested values.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	9	7	2	3	4	1	6	5

a) What is $f(7)$?

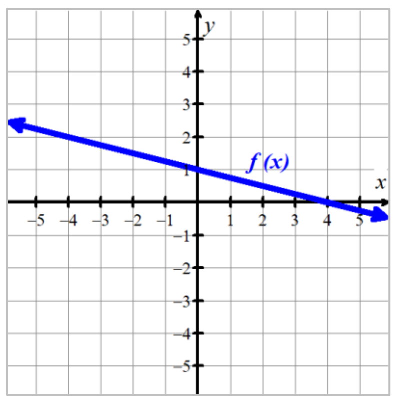
b) If $f(x)=3$, then what is x?

c) What is $f^{-1}(1)$?

d) If $f^{-1}(x)=5$, then what is x?

11. Use the following graph of the function $f(x)$ to determine the requested values.

Figure Ex1.5. 3



a) What is $f(0)$?

b) If $f(x)=0$, then what is x?

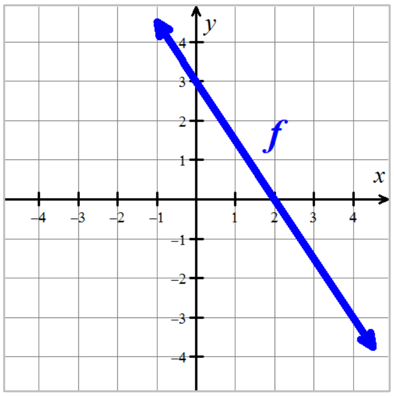
c) What is $f^{-1}(0)$?

d) If $f^{-1}(x)=0$, then what is x?

In Exercises 12 – 15, sketch the graph of the inverse function.

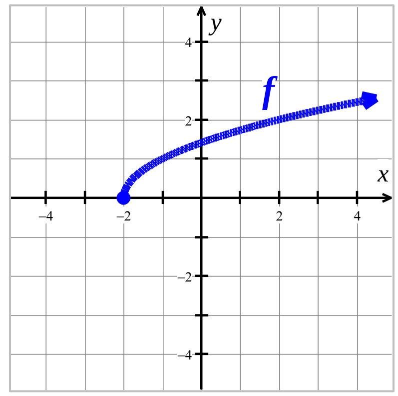
12.

Figure Ex1.5. 4



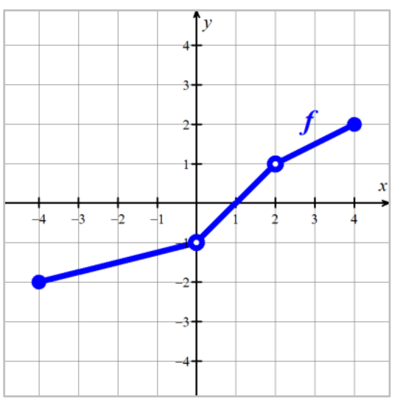
13.

Figure Ex1.5. 5



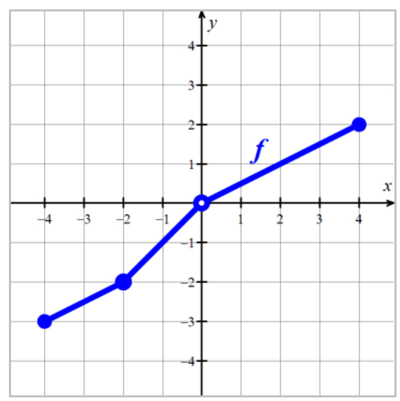
14.

Figure Ex1.5. 6



15.

Figure Ex1.5. 7



In Exercises 16 – 25, show that the given function is one-to-one and find its inverse. Check your answer algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

16. $f(x) = 6x - 2$

17. $f(x) = 42 - x$

18. $f(x) = \dfrac{x-2}{3} + 4$

19. $f(x) = 1 - \dfrac{4+3x}{5}$

20. $f(x) = \sqrt{3x-1} + 5$

21. $f(x) = 2 - \sqrt{x-5}$

22. $f(x) = 3\sqrt{x-1} - 4$

23. $f(x) = 1 - 2\sqrt{2x+5}$

24. $f(x) = 3(x+4)^2 - 5, \; x \le -4$

25. $f(x) = \dfrac{3}{4-x}$

In Exercises 26 – 35, find the inverse of the given one-to-one function.

26. $f(x) = \sqrt[5]{3x-1}$

27. $f(x) = 3 - \sqrt[3]{x-2}$

28. $f(x) = x^2 - 10x,\ x \ge 5$

29. $f(x) = x^2 - 6x + 5,\ x \le 3$

30. $f(x) = 4x^2 + 4x + 1,\ x < -1$

31. $f(x) = \dfrac{x}{1-3x}$

32. $f(x) = \dfrac{2x-1}{3x+4}$

33. $f(x) = \dfrac{4x+2}{3x-6}$

34. $f(x) = \dfrac{-3x-2}{x+3}$

35. $f(x) = \dfrac{x-2}{2x-1}$

36. If $f(x) = \dfrac{x+5}{x+6}$, then what is $f^{-1}(-5)$? How can you determine the value of $f^{-1}(-5)$ without finding $f^{-1}(x)$?

With the help of your classmates, find the inverses of the functions in Exercises 37 – 40.

37. $f(x) = a\,x + b,\ a \ne 0$

38. $f(x) = a\sqrt{x-h} + k,\ a \ne 0,\ x \ge h$

39. $f(x) = a\,x^2 + b\,x + c$ where $a \ne 0,\ x \ge -\dfrac{b}{2a}$

40. $f(x) = \dfrac{a\,x+b}{c\,x+d}$ (See **Exercise 46** below.)

41. Show that the Fahrenheit to Celsius conversion function, $C(F) = \dfrac{5}{9}(F-32)$, is invertible and that its inverse is $F(C) = \dfrac{9}{5}C + 32$, the Celsius to Fahrenheit conversion formula.

42. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

43. The circumference C of a circle is a function of its radius r given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

44. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

45. What graphical feature must a function f possess for it to be its own inverse?

46. What conditions must you place on the values of a, b, c, and d in **Exercise 40** in order to guarantee that the function is invertible?

CHAPTER 2
POLYNOMIAL FUNCTIONS

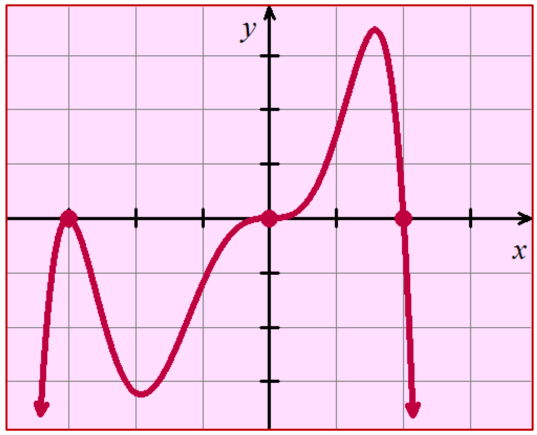


Figure 2.0. 1

Chapter Outline

Introduction

Chapter 2 examines polynomial functions and their graphs using a variety of methods. Throughout the chapter, you will learn many strategies for working with polynomial functions. These strategies lay a foundation for future work, not only with polynomial functions, but with other functions as well. By the end of this chapter, you should be able to move fluidly among representations of polynomial functions and/or use information from one representation to gain information in another.

In Section 2.1, we start by reviewing what you know about quadratic equations in the form of $ax^2+bx+c=0$ (where $a \neq 0$ and *a*, *b*, and *c* are real numbers) including finding solutions by factoring, completing the square, or using the quadratic formula. You will find both real and complex solutions to quadratic equations and explore how the different methods for finding solutions are related to graphing quadratic functions.

Section 2.2 introduces polynomial functions of degree three and higher. It starts by familiarizing you with terms such as degree, leading term, leading coefficient, and constant term. You then explore ideas related to the Intermediate Value Theorem and build on those understandings to find solutions of polynomial equations and their multiplicities. You also explore end behavior of polynomial functions numerically and by considering their leading terms. Armed with understandings about zeros, multiplicities of zeros, x- and y-intercepts, and end behavior, you learn to sketch rough graphs of polynomial functions, primarily in factored form. (Later you will learn to do this when the polynomial is not factored for you.)

Section 2.3 is devoted to helping you gain skills to factor polynomials of degree three or larger, and to use that information for finding solutions of polynomial equations, for finding x-intercepts, and for graphing polynomials. The section starts by developing skills with polynomial long division. You should relate division with polynomials to division with whole numbers; the process is parallel. Just like division with whole numbers, if the remainder is zero, the divisor is a factor; otherwise, the divisor is not a factor. You then extend that understanding to the more efficient process of synthetic division. Vocabulary may become an issue in this section; you should take care to understand how the terms ‘factors’, ‘solutions’, ‘zeros’, and ‘x-intercepts’ are related to each other and to the graph of a polynomial function.

In sections 2.4 and 2.5, you put the ideas learned in the previous three sections together to work with, and understand, polynomials of degree 3 and larger in non-factored form. In 2.4, you will work with polynomial equations with all real solutions, whereas in 2.5, some (or all) of the solutions may be complex. The primary goal of these sections (and the entire chapter) is not to turn you into a graphing calculator (though facility with graphing is very important), but rather to help you gain a deep understanding of polynomial behavior on which you will be able to rely in future mathematics courses.

Section 2.6 focuses on solving inequalities involving polynomials. The section explores both analytical and graphical methods for solving inequalities. The section also gives you opportunities to revisit ideas of multiplicity of roots. By the end of this section, you should gain an appreciation of how fluidity with graphing is a useful tool in solving/understanding polynomial inequalities. The ideas developed in this section for solving inequalities will be extended later with rational functions.

2.1 Quadratic Functions

Learning Objectives

- Graph a quadratic function though transformations of $f(x)=x^2$.
- Change a quadratic function from general to standard form.
- Find the vertex and axis of symmetry of a quadratic function.
- Find the intercepts of a quadratic function.
- Graph a quadratic function using vertex, axis of symmetry and intercepts.
- Solve applications that require finding the maximum or minimum value of a quadratic function.
- Solve equations quadratic in form.

We begin this section with the Quadratic Formula, an important tool that you have likely seen before.

Equation 2.1. The Quadratic Formula: If a, b, and c are real numbers with $a \neq 0$, then the solutions to $ax^2+bx+c=0$ are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 2.1.1. Determine the solution(s) of the equation $2x^2+5x-3=0$.

Solution. We may apply the Quadratic Formula here, noting that $a=2$, $b=5$, and $c=-3$.

$$\begin{aligned} x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &=\frac{-5\pm\sqrt{(5)^2-4(2)(-3)}}{2(2)} \\ &=\frac{-5\pm\sqrt{49}}{4} \\ &=\frac{-5\pm 7}{4} \end{aligned}$$

We have $x=\frac{-5+7}{4}$ and $x=\frac{-5-7}{4}$. After simplifying, we find the solutions are $x=\frac{1}{2}$ and $x=-3$.

□

The discriminant, which can be found within the Quadratic Formula, is useful in identifying the number and type of solutions to a quadratic equation. A definition follows.

Definition 2.1. If a, b, and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

By thinking about the consequences of taking the square root of the discriminant, along with the position of the discriminant in the Quadratic Formula, we have the following.

Determining the Number of Real Solutions to a Quadratic Equation

Let a, b, and c be real numbers with $a \neq 0$.

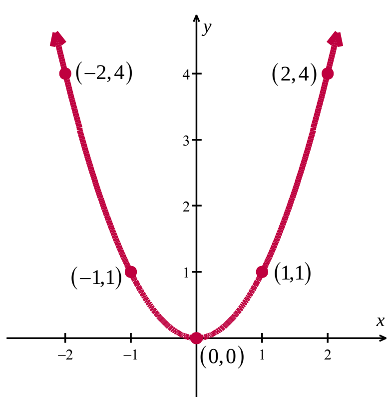
- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has one real solution.[1]
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has two real solutions.

Note that in **Example 2.1.1**, the discriminant was 49, which is greater than zero, and we did indeed have two real solutions: $x = \frac{1}{2}$ and $x = -3$. Knowing the Quadratic Formula and paying attention to the discriminant will be useful throughout this chapter.

Graphing Quadratic Functions through Transformations

The most basic quadratic function is $f(x) = x^2$, which you may recognize as the toolkit function referred to as the squaring function.

Figure 2.1. 1



$f(x) = x^2$

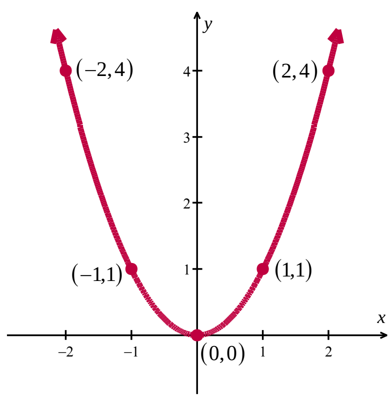
[1] In this case, technically, we have two real but equal solutions.

The shape of this graph should look familiar. It is referred to as a **parabola**. The point $(0,0)$ is called the **vertex** of the parabola. In this case, the vertex is a local minimum and it is also where the absolute minimum value of f can be found. Knowing the graph of $f(x)=x^2$ allows us to graph other quadratic functions using transformations.

Example 2.1.2. Graph the function $g(x)=(x+2)^2-3$, starting with the graph of $f(x)=x^2$ and applying transformations.

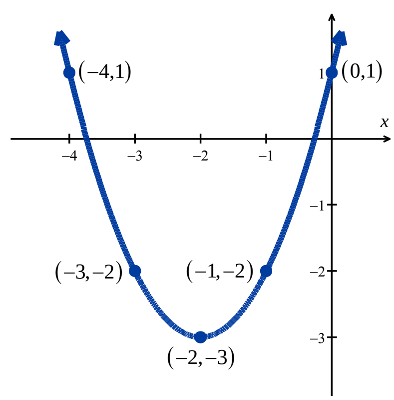
Solution. Since $g(x)=(x+2)^2-3=f(x+2)-3$, we can accomplish this in two steps. First, we subtract 2 from each of the *x*-values of the points on $y=f(x)$. This shifts the graph of $y=f(x)$ to the left 2 units. Next, we subtract 3 from each of the *y*-values of these new points. This moves the graph down 3 units.

Figure 2.1. 2



$f(x)=x^2$

Figure 2.1. 3



$g(x)=(x+2)^2-3$

We see that the vertex $(0,0)$ of the graph of $y=f(x)$ moves to the point $(-2,-3)$ on the graph of $y=g(x)$.

☐

A few remarks about **Example 2.1.2** are in order. We could convert $g(x)$ into a 'simpified' form by expanding and collecting like terms and writing in descending powers of the variable x. Doing so, we find $g(x)=(x+2)^2-3=x^2+4x+1$. This 'simplified' form of $g(x)$ is referred to as the **general form**. We note that a quadratic function in general form does not lend itself easily to graphing via transformations. For that reason, the form of g presented in **Example 2.1.2** is often preferred and is referred to as the **standard form**, or **vertex form**.

Definition 2.2. A function $f(x)=ax^2+bx+c$, where a, b, and c are real numbers with $a\neq 0$, is called a **quadratic function**.

- If a quadratic function is written in the form $f(x)=a(x-h)^2+k$, for some real numbers h and k, we say that f is in **standard form,** or **vertex form**.
- If a quadratic function is written in the form $f(x)=ax^2+bx+c$, we say that f is in **general form**.

As in the above discussion, by simplifying $f(x)=a(x-h)^2+k$, we can rewrite f in general form. Later, we will discuss converting a quadratic function from its general form to its standard form.

The Standard Form of a Quadratic Function

For a quadratic function that is written in standard form, as mentioned and demonstrated earlier, we may graph the function through transformations of the squaring function. A discussion of using transformations to graph $f(x)=a(x-h)^2+k$ follows.

- We start with $g(x)=x^2$ and define $g_1(x)=a\cdot g(x)=a\cdot x^2$. This results in a vertical scaling and/or reflection.[2] Since we multiply the output of g by a, we multiply the *y*-coordinates on the graph of g by a, so the vertex $(0,0)$ of the graph of $g(x)=x^2$ remains $(0,0)$ on the graph of $g_1(x)=ax^2$. The graph of $g_1(x)=ax^2$ opens upward, like the graph of $g(x)=x^2$, if $a>0$, and the graph of $g_1(x)=ax^2$ opens downward if $a<0$.
- Next, we define $g_2(x)=g_1(x-h)=a(x-h)^2$. This induces a horizontal shift right or left h units[3] and moves the point $(0,0)$ on the graph of $g_1(x)=ax^2$ to the point $(h,0)$ on the graph of $g_2(x)=a(x-h)^2$.
- Finally, $f(x)=g_2(x)+k=a(x-h)^2+k$, which causes a vertical shift up or down k units,[4] resulting in moving the point $(h,0)$ on the graph of $g_2(x)=a(x-h)^2$ to the point (h,k) on the graph of $f(x)=a(x-h)^2+k$.

[2] Just a scaling if $a>0$. If $a<0$, there is a reflection involved.
[3] Right if $h>0$, left if $h<0$.
[4] Up if $k>0$, down if $k<0$.

Figure 2.1. 4

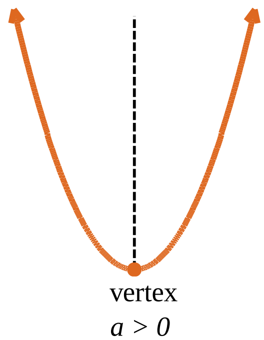
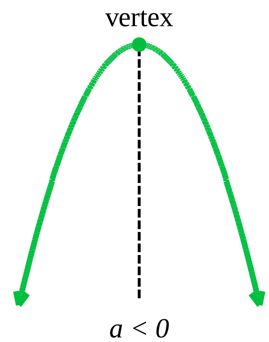


Figure 2.1. 5



Graphs of $f(x)=a(x-h)^2+k$

The point (h,k) is called the **vertex** of the graph of $y=a(x-h)^2+k$. The vertex is the lowest point of the graph if $a>0$ and is the highest point of the graph if $a<0$. Moreover, the symmetry enjoyed by the graph of $y=x^2$ about the *y*-axis is translated to a symmetry about the vertical line $x=h$, which is the vertical line through the vertex.[5] This is called the **axis of symmetry** of the parabola and is dashed in the above figures.

Changing from General Form to Standard Form

Without a doubt, the standard form of a quadratic function allows us to list the attributes of the graphs of such functions quickly and elegantly. What remains to be shown, however, is the fact that every quadratic function can be written in standard form. To convert a quadratic function given in general form into standard form, we employ the ancient rite of 'completing the square'. We remind the reader how this is done in our next example.

Example 2.1.3. Convert the functions below from general form to standard form. Find the vertex, the axis of symmetry, and any *x*- or *y*-intercepts. Graph each function and determine its range.

1. $f(x)=x^2-4x+3$
2. $g(x)=6-x-x^2$

Solution.

1. To convert $f(x)=x^2-4x+3$ from general form to standard form, we complete the square on x^2-4x by first taking half of -4 to get $\frac{1}{2}(-4)=-2$. This tells us that the target perfect square

[5] You should use transformations to verify this!

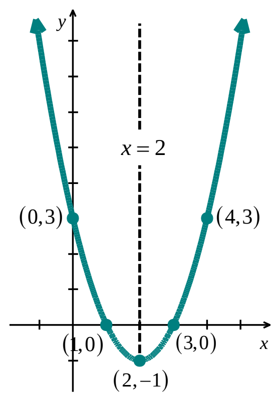
quantity is $(x-2)^2$. To get an expression equivalent to $(x-2)^2$, we need to add $(-2)^2=4$ to x^2-4x to create a perfect square trinomial, but to keep the balance we must also subtract 4.

$$\begin{aligned} f(x) &= x^2-4x+3 \\ &= \left(x^2-4x+4-4\right)+3 \\ &= \left(x^2-4x+4\right)-4+3 \quad \text{group the perfect square trinomial} \\ &= (x-2)^2-1 \quad \text{factor the perfect square trinomial} \end{aligned}$$

In the form $f(x)=(x-2)^2-1$, we readily find the vertex to be $(2,-1)$, and determine that the axis of symmetry is $x=2$.

To find the *x*-intercepts, we set $y=f(x)=0$. We have the choice of two formulas for $f(x)$. Since we recognize $f(x)=x^2-4x+3$ to be easily factorable, we proceed to solve $x^2-4x+3=0$. Factoring gives $(x-3)(x-1)=0$ so that $x=3$ or $x=1$, giving us *x*-intercepts of $(1,0)$ and $(3,0)$. To find the *y*-intercept, we set $x=0$. Once again, the general form $f(x)=x^2-4x+3$ is easiest to work with, and we find $y=f(0)=3$. Hence, the *y*-intercept is $(0,3)$ and, finding its 'mirror image' about $x=2$, the axis of symmetry, we identify an additional point as being $(4,3)$. Putting all of this information together results in the following graph.

Figure 2.1. 6



$$f(x)=x^2-4x+3$$

We see that the range of f is $[-1,\infty)$ and we are done.

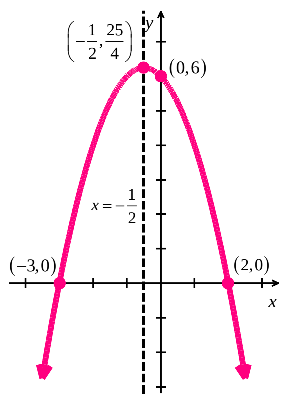
2. To get started, we rewrite $g(x)=6-x-x^2$ as $g(x)=-x^2-x+6$ and note that the coefficient of x^2 is –1, not 1. This means that our first step is to factor out the -1 from both the x^2 and x terms. We then follow the completing the square recipe as before.

$$
\begin{aligned}
g(x) &= -x^2 - x + 6 \\
&= (-1)\left(x^2 + x\right) + 6 \\
&= (-1)\left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right) + 6 && \text{add and subtract } \left(\frac{1}{2}\right)^2 = \frac{1}{4} \text{ to / from } x^2 + x \\
&= (-1)\left(x^2 + x + \frac{1}{4}\right) + (-1)\left(-\frac{1}{4}\right) + 6 && \text{group the perfect square trinomial} \\
&= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4} && \text{factor the perfect square trinomial and simplify}
\end{aligned}
$$

From $g(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}$, we get the vertex of $\left(-\frac{1}{2}, \frac{25}{4}\right)$ and the axis of symmetry $x = -\frac{1}{2}$.

To find the x-intercepts, we set the given function, $g(x) = 6 - x - x^2$, equal to zero. Solving $6 - x - x^2 = 0$, we get $x = -3$ and $x = 2$, so the x-intercepts are $(-3,0)$ and $(2,0)$. Setting $x = 0$, we find $g(0) = 6$ for a y-intercept of $(0,6)$. Plotting these points gives us the following graph.

Figure 2.1. 7



$g(x) = 6 - x - x^2$

We see that the range of g is $\left(-\infty, \frac{25}{4}\right]$.

□

With **Example 2.1.3** fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with $f(x) = ax^2 + bx + c$, assume $a \neq 0$, and complete the square.

$$
\begin{aligned}
f(x) &= ax^2+bx+c \\
&= a\left(x^2+\frac{b}{a}x\right)+c && \text{factor out the coefficient of } a \text{ from } x^2 \text{ and } x \\
&= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right)+c && \text{add and subtract } \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 \text{ to / from } x^2+\frac{b}{a}x \\
&= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+a\left(-\frac{b^2}{4a^2}\right)+c && \text{group the perfect square trinomial} \\
&= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c && \text{factor and simplify} \\
&= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} && \text{obtain common denominator}
\end{aligned}
$$

Comparing this last expression with the standard form, we identify $(x-h)$ with $\left(x+\frac{b}{2a}\right)$ so that $h=-\frac{b}{2a}$. Instead of memorizing the value $k=\frac{4ac-b^2}{4a}$, we note that (h,k) is on the graph of f, and so $f(h)=k$, giving us $f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a}$. As such, we have derived a vertex formula for the general form. We next summarize both vertex formulas.

Equation 2.2. Vertices of Quadratic Functions: Suppose a, b, c, h, and k are real numbers with $a\neq 0$.

- If $f(x)=a(x-h)^2+k$, the vertex of the graph of $y=f(x)$ is the point (h,k).
- If $f(x)=ax^2+bx+c$, the vertex of the graph of $y=f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

We next incorporate the information from **Equation 2.2** into a general strategy for graphing parabolas that are presented to us in either general form or standard form.

To Graph the Parabola $f(x)=ax^2+bx+c$ or $f(x)=a(x-h)^2+k$ with $a\neq 0$:

1. Determine if the graph opens upward (a is positive) or downward (a is negative).
2. Find the vertex, $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$ or (h,k), and the axis of symmetry, $x=-\dfrac{b}{2a}$ or $x=h$.
3. Find all *x*- and *y*-intercepts.
4. Find additional points, if necessary. (For example, when there are no *x*-intercepts.)
5. Plot the points from the previous steps and draw the axis of symmetry.
6. Connect the points with a smooth curve and extend the curve to complete your graph.

In the next example, we apply our graphing strategy to a parabola of the form $f(x)=ax^2+bx+c$.

Example 2.1.4. Graph the function $f(x)=x^2+4x+7$ after finding the vertex, the axis of symmetry, and any *x*- or *y*-intercepts.

Solution. We begin by noting that $a=1>0$ and so the parabola will open upwards. Next, we identify the *x*-coordinate of the vertex.

$$\begin{aligned}-\frac{b}{2a}&=-\frac{4}{2(1)}\\&=-2\end{aligned}$$

Substituting $x=-2$ into $f(x)=x^2+4x+7$, we find the second coordinate of the vertex.

$$\begin{aligned}f(-2)&=(-2)^2+4(-2)+7\\&=3\end{aligned}$$

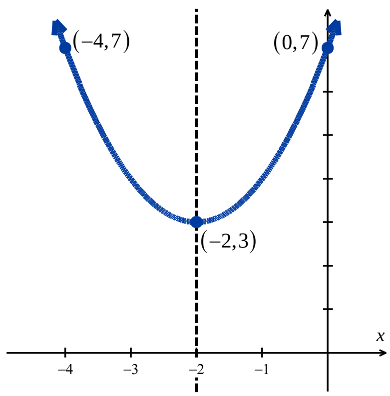
From the vertex of $(-2,3)$, we see that the axis of symmetry is $x=-2$.

To find the *x*-intercpts, we set $f(x)=x^2+4x+7=0$. Noting that x^2+4x+7 is not easily factorable, we use the Quadratic Formula. With $a=1$, $b=4$, and $c=7$, we have

$$\begin{aligned}x&=\frac{-4\pm\sqrt{(4)^2-4\cdot1\cdot7}}{2\cdot1}\\&=\frac{-4\pm\sqrt{-12}}{2}\end{aligned}$$

Since the discriminant, -12, is less than zero, $x^2+4x+7=0$ does not have any real solutions. Thus, there are no *x*-intercepts.[6] To find the *y*-intercept, we set $x=0$ and find $f(0)=(0)^2+4(0)+7=7$ for a *y*-intercept of $(0,7)$. Reflecting the point $(0,7)$ about the axis of symmetry, $x=-2$, gives us the additional point $(-4,7)$. The graph appears below.

Figure 2.1. 8



$f(x)=x^2+4x+7$

□

For accuracy in sketching the curve it is helpful to plot additional points, such as the points $(-3,4)$ and $(-1,4)$ in the preceding example.

Solving Applications of Quadratic Functions

Example 2.1.5. The weekly profit, in dollars, made by selling x PortaBoy Game Systems is $P(x)=-1.5x^2+170x-150$ with the restriction that $0 \le x \le 166$.

1. Graph $y=P(x)$. Include the *x*- and *y*-intercepts as well as the vertex and axis of symmetry.
2. Interpret the zeros of P.
3. Interpret the vertex of the graph of $y=P(x)$.

Solution.

1. To find the *x*-intercepts, we set $P(x)=0$ and solve $-1.5x^2+170x-150=0$ with the aid of the quadratic formula.

[6] Since the vertex is above the *x*-axis and the parabola opens upward, there are obviously no *x*-intercepts. Had we made this observation first, we could have skipped our attempt to calculate the *x*-intercepts.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{(-1)\left(170 \mp 20\sqrt{70}\right)}{(-1)(3)} \\ &= \frac{170 \mp 20\sqrt{70}}{3} \end{aligned}$$

We get two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$. To find the y-intercept, we set $x=0$ and find $y=P(0)=-150$ for a y-intercept of $(0,-150)$.

To find the vertex, we use the fact that $P(x)=-1.5x^2+170x-150$ is in the general form of a quadratic function and appeal to **Equation 2.2**. We first determine the value of the x-coordinate.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{170}{2(-1.5)} \\ &= \frac{170}{3} \end{aligned}$$

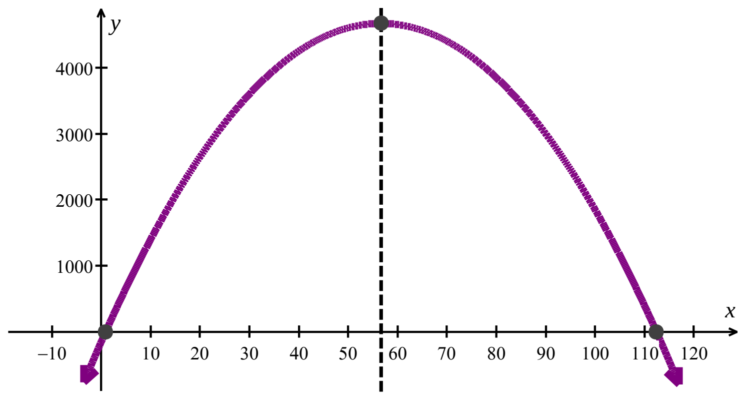
We next compute the y-coordinate of the vertex.

$$\begin{aligned} P\left(\frac{170}{3}\right) &= -1.5\left(\frac{170}{3}\right)^2 + 170\left(\frac{170}{3}\right) - 150 \\ &= -\frac{3}{2}\cdot\frac{28900}{9} + \frac{28900}{3} - 150 \\ &= -\frac{14450}{3} + \frac{28900}{3} - \frac{450}{3} \\ &= \frac{14000}{3} \end{aligned}$$

Our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x=\frac{170}{3}$.

To sketch a reasonable graph, we approximate the x-intercepts, $(0.89,0)$ and $(112.44,0)$, and the vertex, $(56.67,4666.67)$. We adjust the scales on the x-axis and the y-axis so that intercepts and vertex are visible.

Figure 2.1. 9



$y = P(x)$

2. The zeros of P are the solutions to $P(x)=0$, which we have found to be approximately 0.89 and 112.44. We see from the graph that as long as x is between 0.89 and 112.44, the graph $y = P(x)$ is above the x-axis, meaning $y = P(x) > 0$. This tells us that for these values of x, a profit is being made. Since x represents the weekly sales of PortaBoy Game Systems, we round the zeros to the positive integers 1 and 112, respectively. Then, as long as at least 1 game system, but no more than 112 systems, are sold weekly, the retailer will make a profit.

3. From the graph, we see that the maximum value of P occurs at the vertex, which is approximately $(56.67,4666.67)$. As above, x represents the weekly sales of PortaBoy systems, so we cannot sell 56.67 game systems. Comparing $P(56)=4666$ and $P(57)=4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

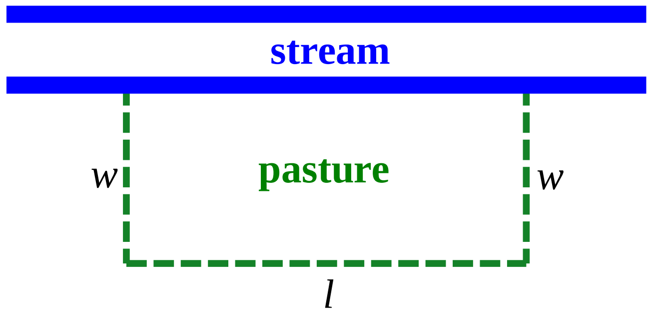
□

Our next example is a classic application of quadratic functions.

Example 2.1.6. Much to Kyle's surprise and delight, he inherits a large parcel of land in Wasatch County from one of his (e)strange(d) relatives. The time is finally right for him to pursue his dream of farming alpacas. He wishes to build a rectangular pasture, and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a stream (so no fencing is required on that side), what are the dimensions of the pasture with the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpacas can Kyle keep in his pasture?

Solution. It is always helpful to sketch the problem situation, as we do below.

Figure 2.1. 10



We are tasked to find the dimensions of the pasture that would give a maximum area. We let w denote the width of the pasture and we let l denote the length of the pasture. The area of the pasture, which we will call A, is related to w and l by the equation $A = w \cdot l$. Since our objective is to maximize A, we refer to $A = w \cdot l$ as the **objective function**.

We are given that the total amount of fencing available is 200 feet, which means $w + l + w = 200$, or $l + 2w = 200$. This equation limits the possibilities for dimensions and area, and is thus referred to as a **constraint**.

In order to use the tools given to us in this section to maximize A, we need to write A as a function of just one variable, either w or l. This is where we use the constraint $l + 2w = 200$. Solving for l, we find $l = 200 - 2w$, and we substitute this into our objective function, $A = l \cdot w$.

$$\begin{aligned} A &= (200 - 2w)(w) \\ &= 200w - 2w^2 \\ &= -2w^2 + 200w \end{aligned}$$

Before we go any further, we need to find the applied domain of A so that we know what values of w make sense in this problem situation. Since w represents the width of the pasture, $w > 0$. Likewise, l represents the length of the pasture so $l = 200 - 2w > 0$. Solving this inequality, we find $w < 100$. Hence, the function we wish to maximize is $A(w) = -2w^2 + 200w$ for $0 < w < 100$.

Since A is a quadratic function (of w), the graph of $y = A(w)$ is a parabola. With the coefficient of w^2 being -2, we know that this parabola opens downward. Thus, there is a maximum value to be found, and that maximum value occurs at the vertex. We use the vertex formula to find w.

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{200}{2(-2)} \\ &= 50 \end{aligned}$$

Since $w = 50$ lies in the applied domain, $0 < w < 100$, the area of the pasture is maximized when the width is 50 feet. To find the corresponding length, we use $l = 200 - 2w$, from which $l = 200 - 2(50) = 100$. Thus, the maximum area occurs when the width is 50 feet and the length is 100 feet. To determine the maximum area, we can set $w = 50$ in $A(w) = -2w^2 + 200w$:

$$\begin{aligned} A(50) &= -2(50)^2 + 200(50) \\ &= 5000 \end{aligned}$$

Thus, the maximum area is 5000 square feet. If an average alpaca requires 25 square feet of pasture, Kyle can raise $\frac{5000}{25} = 200$ average alpacas.

□

Solving Equations Quadratic in Form

Throughout this textbook, you will find equations that may be rewritten in the form $au^2 + bu + c = 0$ where u is an expression of the variable we are solving for. Such equations are referred to as **quadratic in form**. Strategies for finding solutions to quadratic equations can be extended to solving equations that are quadratic in form.

Example 2.1.7. Solve the equation $5x^6 = 4 - 8x^3$.

Solution. It is often helpful when solving a non-linear equation to reorganize the equation so that we have an expression equal to zero.

$$\begin{aligned} 5x^6 &= 4 - 8x^3 \\ 5x^6 + 8x^3 - 4 &= 0 \end{aligned}$$

Noting that $x^6 = (x^3)^2$, we may rewrite the equation in quadratic form. We let $u = x^3$ to get

$$\begin{aligned} 5(x^3)^2 + 8(x^3) - 4 &= 0 \\ 5u^2 + 8u - 4 &= 0 \end{aligned}$$

The equation $5u^2+8u-4=0$ is now in a form we know how to solve. Because this quadratic is easily factorable, we will solve in this manner. However, completing the square or applying the quadratic formula may also be used.

$$\begin{aligned}5u^2+8u-4&=0\\(5u-2)(u+2)&=0\end{aligned}$$

Thus, $u=\frac{2}{5}$ or $u=-2$. However, recall that $5u^2+8u-4=0$ is not the original equation, it is the equation we got by substituting $u=x^3$. We still need to solve for x, so there is one more step. For $x^3=\frac{2}{5}$, we solve for x to find $x=\sqrt[3]{\frac{2}{5}}$, and for $x^3=-2$ we have $x=\sqrt[3]{-2}$.

□

Example 2.1.8. Solve the equation $x^{\frac{1}{2}}-x^{\frac{1}{4}}=6$.

Solution. Again we notice that we can rearrange the equation to set one side equal to zero, and we see that the highest power term is the square of the next highest power term.

$$\begin{aligned}x^{\frac{1}{2}}-x^{\frac{1}{4}}&=6\\x^{\frac{1}{2}}-x^{\frac{1}{4}}-6&=0\\\left(x^{\frac{1}{4}}\right)^2-x^{\frac{1}{4}}-6&=0\end{aligned}$$

We let $u=x^{\frac{1}{4}}$ so that $u^2-u-6=0$. We solve again by factoring, although any method of solving for u may be used.

$$\begin{aligned}u^2-u-6&=0\\(u-3)(u+2)&=0\end{aligned}$$

Thus, $u=3$ or $u=-2$. Substituting x back in, we get $x^{\frac{1}{4}}=3$ or $x^{\frac{1}{4}}=-2$. Let's look at each case individually.

- For $x^{\frac{1}{4}}=3$, noting that the outputs of even roots are non-negative numbers, it is possible to have $x^{\frac{1}{4}}=3$ so we raise both sides to the fourth power to find $x=81$.
- For $x^{\frac{1}{4}}=-2$, there is a problem; in the real numbers, even roots do not result in negative outputs. Thus, there is no solution for $x^{\frac{1}{4}}=-2$.

The only solution is $x=81$. □

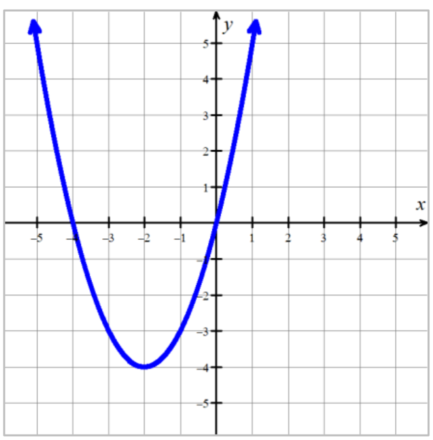
2.1 Exercises

1. How can the vertex of a parabola be used in solving real world problems?
2. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
3. Graph the function $g(x) = (x-3)^2 + 4$ by starting with the graph of $f(x) = x^2$ and applying transformations. Identify the vertex of $g(x)$.
4. Graph the function $g(x) = -2(x+4)^2$ by starting with the graph of $f(x) = x^2$ and applying transformations. Identify the vertex of $g(x)$.

In Exercises 5 and 6, the graph of a quadratic function is given. Identify the equation, in standard form, of the function that has been graphed.

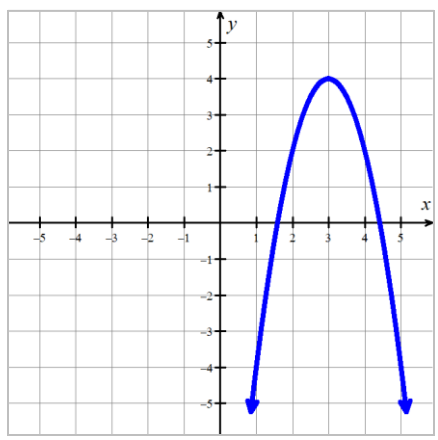
5.

Figure Ex2.1. 1



6.

Figure Ex2.1. 2



In Exercises 7 – 10, convert the function from general form to standard form.

7. $f(x) = x^2 - 2x - 8$
8. $f(x) = x^2 - 6x - 1$
9. $f(x) = 2x^2 - 4x + 7$
10. $f(x) = -4x^2 - 12x + 3$

In Exercises 11 – 26, find the vertex, the axis of symmetry and any *x*- or *y*-intercepts. Graph each function and determine its range.

11. $f(x) = x^2 + 2$
12. $f(x) = -(x+2)^2$
13. $f(x) = -(x+2)^2 + 2$
14. $f(x) = x^2 - 2x$
15. $f(x) = x^2 - 2x - 8$
16. $f(x) = x^2 - 6x - 1$

17. $f(x) = x^2 - 5x - 6$

18. $f(x) = x^2 - 7x + 3$

19. $f(x) = -2(x+1)^2 + 4$

20. $f(x) = 2x^2 - 4x - 1$

21. $f(x) = 2x^2 - 4x + 7$

22. $f(x) = -2x^2 + 5x - 8$

23. $f(x) = 4x^2 - 12x - 3$

24. $f(x) = -3x^2 + 4x - 7$

25. $f(x) = x^2 + x + 1$

26. $f(x) = -3x^2 + 5x + 4$

In Exercises 27 – 32, find the real solution(s) of the given equation.

27. $(x+3)^2 - 4(x+3) - 5 = 0$

28. $\left(\dfrac{1}{x+1}\right)^2 - 2\left(\dfrac{1}{x+1}\right) - 3 = 0$

29. $x - 3\sqrt{x} + 2 = 0$

30. $4x^4 + 9 = 13x^2$

31. $2x^{-2} = x^{-1} + 1$

32. $y^{\frac{1}{3}} + y^{\frac{1}{6}} - 2 = 0$

In Exercises 33 – 37, the profit function $P(x)$ is given.

- Find the number of items that need to be sold in order to maximize profit. Be sure that your answer is reasonable in the context of the problem.
- Find the maximum profit.
- Find and interpret the zeros of $P(x)$.

33. The profit, in dollars, made by selling x "I'd rather be a Sasquatch" t-shirts is $P(x) = -2x^2 + 28x - 26$, $0 \le x \le 15$.

34. The profit, in dollars, made by selling x bottles of 100% all-natural certified free-trade organic Sasquatch Tonic is $P(x) = -x^2 + 25x - 100$, $0 \le x \le 35$.

35. The profit, in cents, made by selling x cups of Mountain Thunder Lemonade at Junior's lemonade stand is $P(x) = -3x^2 + 72x - 240$, $0 \le x \le 30$.

36. The daily profit, in dollars, made by selling x Sasquatch Berry Pies is $P(x) = -0.5x^2 + 9x - 36$, $0 \le x \le 24$.

37. The monthly profit, in hundreds of dollars, made by selling x custom built electric scooters is $P(x) = -2x^2 + 120x - 1000$, $0 \le x \le 70$.

38. Using data from the Bureau of Transportation statistics, the average fuel economy F (in miles per gallon) for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \le t \le 28$, where t is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

39. The temperature T (in degrees Fahrenheit), t hours after 6 AM, is given by $T(t) = -\frac{1}{2}t^2 + 8t + 32$, $0 \le t \le 12$. What is the warmest temperature of the day? When does this happen?

40. Suppose $C(x) = x^2 - 10x + 27$, $3 \le x \le 15$, represents the marginal cost (in hundreds of dollars) to produce an additional x thousand pens. How many additional pens should be produced to minimize the marginal cost? What is the minimum marginal cost?

41. Suppose $h(x) = -\frac{1}{200}x^2 + \frac{4}{5}x + 3$ represents a baseball's height above the ground where x is the baseball's horizontal distance from the home plate. Both x and h are measured in feet. What is the maximum height of the ball and at what distance from the home plate does it occur?

42. Dani wishes to plant a vegetable garden along one side of her house. She wants the plot to be rectangular to simplify landscaping maintenance. In her garage, she has 32 linear feet of fencing. Since one side of the garden will border the house, Dani doesn't need fencing along that side. What are the dimensions of the garden that will maximize the area of the garden? What is the maxium area of the garden?

43. In the situation of **Example 2.1.6**, Kyle has a nightmare that one of his alpacas fell into the stream and was injured. To avoid this, he wants to move his rectangular pasture away from the stream. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions will now maximize the area of the pasture? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpacas can he raise?

44. What is the largest rectangular area one can enclose with 14 inches of string?

45. The height of an object dropped from the roof of an eight story building is modeled by $h(t) = -16t^2 + 64$, $0 \le t \le 2$. Here, h is the height (in feet) of the object above the ground, t seconds after the object is dropped. How long, after being dropped, is it before the object hits the ground?

46. The height h (in feet) of a model rocket above the ground, t seconds after lift-off, is given by $h(t) = -5t^2 + 100t$, $0 \le t \le 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

47. Jason participates in the Highland Games. In one event, the hammer throw, the height h (in feet) of the hammer above the ground t seconds after Jason lets it go is modeled by $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

48. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t (in seconds) of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$, where s is in meters, v_0 (the object's initial velocity) is in meters per second, and s_0 (the initial height of the object) is in meters.

(a) What is the applied domain of this function?

(b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

(c) Come up with a scenario in which $s_0 < 0$.

(d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

(e) Now shoot the marble from the top of a tower that is 25 meters tall. When does the marble hit the ground?

(f) What would the height function be if, instead of shooting the marble up off the top of the tower, you were to shoot it straight DOWN from the top of the tower?

49. The two towers of a suspension bridge are 400 feet apart. The parabolic cable attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

50. Find all points on the line $y = 1 - x$ that are 2 units away from $(1, -1)$.

51. Let L be the line $y = 2x + 1$. Find a function $D(x)$ that measures the distance squared from a point on L to $(0,0)$. Use this to find the point on L closest to $(0,0)$.

52. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros, then the x-coordinate of the vertex is the midpoint of the zeros.

2.2 Graphs of Polynomials

Learning Objectives

- Determine whether or not a function is a polynomial.
- Identify the degree, leading term, leading coefficient, and constant term of a polynomial function.
- Determine the existence of zeros using the Intermediate Value Theorem.
- Find zeros of polynomial functions and their multiplicities; use multiplicity to determine the behavior of the graph at each zero.
- Identify the end behavior of a polynomial function.
- Sketch the graph of a polynomial function using zeros, multiplicities, and end behavior.
- Solve applications that require finding the maximum or minimum value of a polynomial function.

Quadratic functions belong to a much larger group of functions called **polynomial functions**. We begin our formal study of general polynomial functions with a definition and some examples.

Definition 2.3. A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, …, a_n are real numbers and n is a nonnegative integer. The domain of a polynomial function is $(-\infty, \infty)$.

In an effort to understand **Definition 2.3**, we look at an example of a polynomial function, $f(x) = 4x^5 - 3x^2 + 2x - 5$. We can rewrite f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with **Definition 2.3**, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. The subscript on each coefficient, a, merely indicates to which power of x the coefficient belongs.

Functions that are Polynomials

The following example provides some insight into determining whether or not a function represents a polynomial.

Example 2.2.1. Determine if the following functions are polynomials. Explain your reasoning.

1. $g(x)=\sqrt{2}\,x-\pi x+3$
2. $p(x)=\sqrt{2x}-\pi x+3$
3. $q(x)=\dfrac{x^2}{2}+5x$
4. $f(x)=3^x+5x^2$
5. $h(x)=\dfrac{2}{x^2}+7$
6. $j(x)=3$

Solution.

1. We note $g(x)=\sqrt{2}\,x-\pi x+3$ can be written as $g(x)=\left(\sqrt{2}-\pi\right)x+3$. Since $\sqrt{2}-\pi$ is a real number, as is 3, we find that g is of the form $g(x)=a_1x+a_0$ and is therefore a polynomial by **Definition 2.3**.

2. We rewrite $p(x)=\sqrt{2x}-\pi x+3$ as $p(x)=\sqrt{2}\,x^{\frac{1}{2}}-\pi x+3$ and note that in the term $\sqrt{2}\,x^{\frac{1}{2}}$ the power $\dfrac{1}{2}$ is not an integer. Thus, p is not a polynomial.

3. Once again, we start by rewriting $q(x)=\dfrac{x^2}{2}+5x$ to match the format given to us in **Definition 2.3**. We get $q(x)=\dfrac{1}{2}x^2+5x+0$, a polynomial of the form $q(x)=a_2x^2+a_1x+a_0$.

4. We note that the function $f(x)=3^x+5x^2$ has a first term of 3^x. While it may be tempting to think of 3^x as being somehow related to x^3, these two terms are very different. Later on, we will study functions that include terms like 3^x, but for now we simply note that this term does not belong in a polynomial function, and so f is not a polynomial.

5. The function $h(x)=\dfrac{2}{x^2}+7$ can be rewritten as $h(x)=2x^{-2}+7$. This function is not a polynomial since the term $2x^{-2}$ contains a negative power of x. **Definition 2.3** requires powers to be nonnegative integers.

6. The function $j(x)=3$ is of the form $j(x)=a_0$, with $a_0=3$, and is therefore a polynomial.

□

We continue with the introduction of some terminology involving characteristics of polynomials.

Characteristics of Polynomials

Definition 2.4. Suppose f is a polynomial function.

- Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$, $n \geq 1$, we say
  - The **degree** of f is n, the highest power of the variable that appears in the polynomial.
  - The **leading term** of f is $a_n x^n$, the term containing the highest power of the variable.
  - The **leading coefficient** of f is a_n, the coefficient of the leading term.
  - The **constant term** of f is a_0, the term with no variable.
- Given $f(x) = a_0$ with $a_0 \neq 0$, we say the degree of f is zero.

One good thing that comes from **Definition 2.4** is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions. We continue with an example that puts **Definition 2.4** to good use.

Example 2.2.2. Find the degree, leading term, leading coefficient, and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x - x^3$
3. $h(x) = \dfrac{4-x}{5}$
4. $p(x) = (2x-1)^3(x-2)(3x+2)$

Solution.

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of **Definition 2.4** and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4, and the constant term is -5.

2. For $g(x) = 12x - x^3$, we find the highest power of the variable x to be 3, and so the polynomial has degree 3. The leading term is $-x^3$ since it is the term with the power of x being 3. The leading coefficient, or the coefficient of the leading term, is -1. Finally, there is no term without a variable and so the constant term is 0.

3. We start by rewriting h so that it resembles the form given in **Definition 2.4**.

$$\begin{aligned} h(x) &= \frac{4-x}{5} \\ &= \frac{4}{5} - \frac{x}{5} \\ &= -\frac{1}{5}x + \frac{4}{5} \end{aligned}$$

Since the highest power of x is one, the degree of this polynomial is one, its leading term is $-\frac{1}{5}x$, and its leading coefficient is $-\frac{1}{5}$. The constant term is $\frac{4}{5}$.

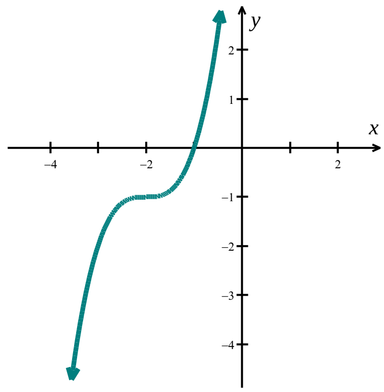
4. It is not necessary to get $p(x) = (2x-1)^3(x-2)(3x+2)$ in the form of **Definition 2.4** since we can glean the information requested about p without multiplying out the entire expression. The leading term of p will be the term that has the highest power of x. The way to get this term is to multiply together the terms with the highest power of x from each factor. Hence, the leading term of p is $(2x)^3(x)(3x) = 24x^5$. This means that the degree of p is five and the leading coefficient is 24.

   As for the constant term, we can implement a similar strategy. The constant term is obtained by multiplying the constant terms from each of the factors: $(-1)^3(-2)(2) = 4$. So, the constant term is 4. Notice that we have shown that if we multiply out the entire expression, in descending powers, we get $p(x) = (2x-1)^3(x-2)(3x+2) = 24x^5 + \cdots + 4$.

□

We turn our attention to graphs of polynomials in general. The toolkit functions that are polynomials include the constant function, the identity function, the squaring function and the cubing function. Transformations of these functions are also polynomials. An example is $f(x) = (x+2)^3 - 1$.

Figure 2.2. 1



$f(x)=(x+2)^3-1$

The graph of $f(x)=(x+2)^3-1$ demonstrates the properties of being **continuous** and **smooth**.[7] The following graphs are **not** continuous and smooth. The first three graphs are not continuous at $x=2$, while the fourth graph is not smooth at $x=2$. To summarize, a continuous function has a one piece graph with no breaks (holes or jumps) and a smooth function has no sharp edges such as corners in its graph.

Figure 2.2. 2

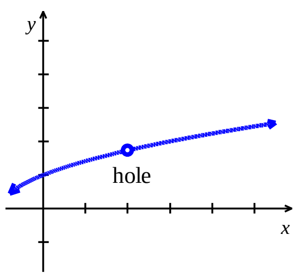


Figure 2.2. 3

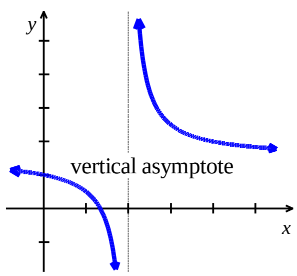


Figure 2.2. 4

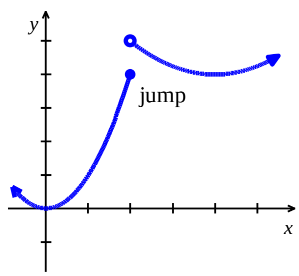
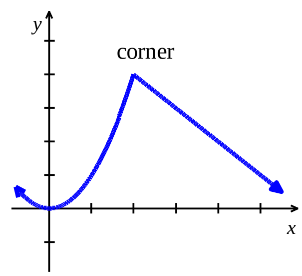


Figure 2.2. 5



None of these four graphs is a polynomial function. Before moving on, we note that graphs of polynomial functions are continuous and smooth everywhere.

The Intermediate Value Theorem

The property of continuous functions having one-piece graphs with no breaks leads to the following theorem.

[7] The terms ‘continuous’ and ‘smooth’ will be discussed more precisely in future mathematics classes.

Theorem 2.1. The Intermediate Value Theorem: Suppose f is a continuous function on an interval containing $x=a$ and $x=b$ with $a<b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x=a$ and $x=b$; that is, for at least one real number c such that $a<c<b$, we have $f(c)=0$.

The Intermediate Value Theorem means that the graph of a continuous function cannot be above the *x*-axis at one point and below the *x*-axis at another point without crossing the *x*-axis somewhere in between. We will not show a formal proof here, but please think about why and how this theorem works before moving on. The following example uses the Intermediate Value Theorem to establish a fact that most students take for granted.

Example 2.2.3. Use the Intermediate Value Theorem to establish that there is a positive number whose square is 2.

Solution. Consider the polynomial function $f(x)=x^2-2$. Then $f(1)=-1$ and $f(3)=7$. Since $f(1)$ and $f(3)$ have different signs and f is a polynomial, so is continuous, the Intermediate Value Theorem guarantees that there is a real number c between 1 and 3 with $f(c)=0$. The number c, being between 1 and 3, is a positive number whose square is 2.

□

Our primary use of the Intermediate Value Theorem is in graphing polynomial functions since it guarantees that a polynomial function is always positive or always negative on intervals that do not contain any of its zeros.

Graphing Polynomials

Example 2.2.4. Sketch a rough graph of the polynomial function $f(x)=(x-2)(x+3)^2$.

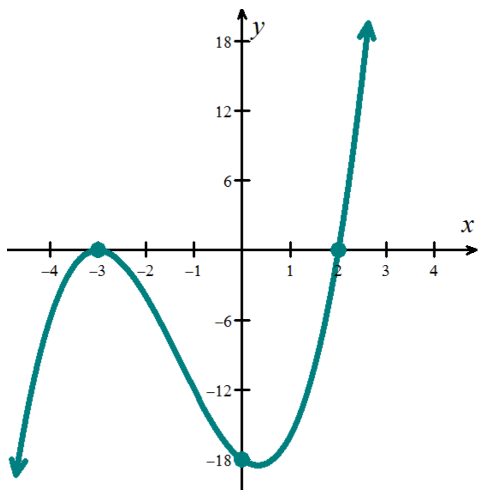
Solution. Our goal in sketching a 'rough' graph is to draw a smooth, continuous curve that includes *x*-intercepts and that shows intervals where the graph is above the *x*-axis and intervals where the graph is below the *x*-axis. First, we find the zeros of f by solving $(x-2)(x+3)^2=0$. We get $x=-3$ and $x=2$. The corresponding x-intercepts, $(-3,0)$ and $(2,0)$, divide the *x*-axis into three intervals: $(-\infty,-3)$, $(-3,2)$, and $(2,\infty)$.

We next test a value in each of these intervals to determine if the function f is positive or negative. Where f is positive, its graph is above the *x*-axis; where f is negative, its graph is below the *x*-axis.

Interval	Test Value	Function Value	Location of Graph
$(-\infty,-3)$	$x=-4$	$f(-4)=-6$	Below *x*-axis
$(-3,2)$	$x=0$	$f(0)=-18$	Below *x*-axis
$(2,\infty)$	$x=3$	$f(3)=+36$	Above *x*-axis

Knowing that the graph is smooth and continuous, we use the *x*-intercepts and intervals where f is above/below the *x*-axis to sketch a graph of the function. To include the *y*-intercept in our graph, we set $x=0$ to find $y=-18$. The point $(0,-18)$ may or may not be the local minimum point; the following is a rough graph.

Figure 2.2. 6



A sketch of $y=f(x)$

□

In **Example 2.2.4**, if we took the time to find the leading term of f, we would find it to be x^3. We have yet to look at the end behavior of polynomial functions, but we will find that the leading term gives us important information about the direction of the graph's tails.

The Multiplicity of a Zero

Before discussing end behavior, we look at the multiplicity of the zero of a polynomial and its effect on the graph. If a polynomial is written as a product of linear factors, then the multiplicity of a zero would be the number of occurrences of the factor corresponding to that zero. From **Example 2.2.4**, we have $f(x)=(x-2)(x+3)(x+3)$. Since $(x-2)$ occurs one time, $x=2$ is a zero of multiplicity one. The factor $(x+3)$ occurs twice, and so $x=-3$ is a zero of multiplicity two.

We note that $(x-2)$ changes from negative to positive at $x=2$, resulting in a change of sign for the function f. On the other hand, $(x+3)^2$ remains positive on both sides of $x=-3$, so there is not a change in sign for the function f at that point. The effects of multiplicity are summarized below.

The Role of Multiplicity

Suppose f is a polynomial function and $x=c$ is a real zero of f with multiplicity m.

- If m is even, the graph of $y=f(x)$ touches, but does not cross, the *x*-axis at $(c,0)$.
- If m is odd, the graph of $y=f(x)$ crosses through the *x*-axis at $(c,0)$.

The End Behavior of Graphs of Polynomials

We start by observing the behavior of the graphs of $f(x)=x^2$, $h(x)=x^6$, $g(x)=-x^2$ and $j(x)=-x^6$. We note that these graphs have similar shapes. However, as the degree increases, the graphs flatten somewhat near the origin and become steeper away from the origin. Additionally, the graphs of g and j are reflections of the graphs of f and h, respectively, about the *x*-axis and thus have tails that head off in directions opposite to those of f and h.

Figure 2.2. 7

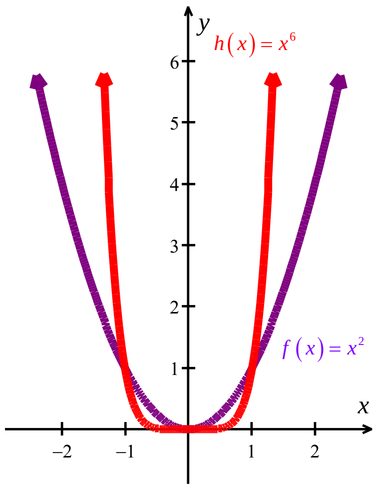
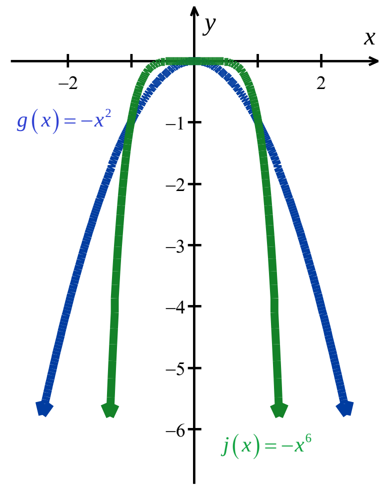


Figure 2.2. 8



The **end behavior** of a function is a way to describe what is happening to the function values (the *y*-values) as the *x*-values approach the 'ends' of the *x*-axis.[8] That is, the end behavior is what is happening to *y*-values for increasingly larger negative *x*-values (written 'as $x \to -\infty$' and read 'as x approaches negative infinity') and, on the flip side, for increasingly larger positive *x*-values (written 'as $x \to \infty$' and read 'as x approaches infinity').

For example, given $f(x)=x^2$, as $x \to -\infty$, we imagine substituting $x=-100$, $x=-1000$, etc., into f to get $f(-100)=10000$, $f(-1000)=1000000$, and so on. Thus, the function values are becoming larger and larger positive numbers, without an upper bound. To describe this behavior, we write

$$\text{as } x \to -\infty, \quad f(x) \to \infty$$

If we study the behavior of f as $x \to \infty$, we see that, in this case too, $f(x) \to \infty$. The same can be said for any function of the form $f(x)=x^n$ where n is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the *x*-axis, we have the following.

[8] Of course, there are no ends to the *x*-axis.

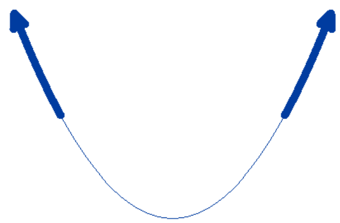
End Behavior of Functions $f(x) = a x^n$, n even

Suppose $f(x) = a x^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

- For $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- For $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

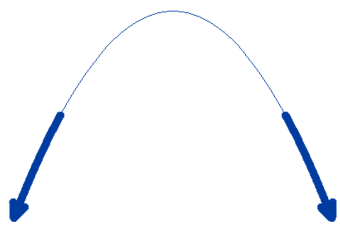
Graphically:

Figure 2.2. 9



$a > 0$

Figure 2.2. 10



$a < 0$

We now turn our attention to functions of the form $f(x) = x^n$ where n is an odd natural number. Following are the graphs of $f(x) = x^3$, $h(x) = x^7$, $g(x) = -x^3$ and $j(x) = -x^7$. The flattening and steepening that we saw with the even powers presents itself here as well, for $n \geq 3$.

Figure 2.2. 11

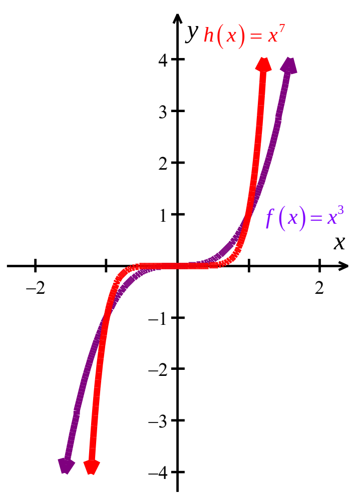
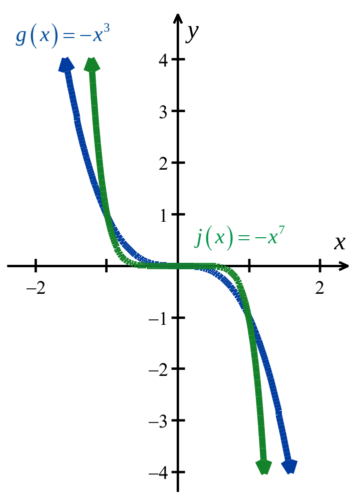


Figure 2.2. 12



The end behavior of the functions f and h is the same: $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$. We note that the graphs of the functions g and j result from reflections of the graphs of f and h, respectively, about the *x*-axis, resulting in 'opposite' end behavior. As with the even degreed functions, we can generalize their end behavior.

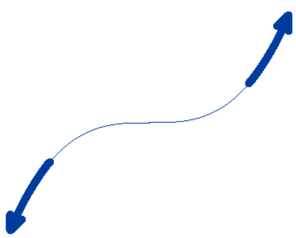
End Behavior of Functions $f(x)=ax^n$, n odd

Suppose $f(x)=ax^n$ where $a\neq 0$ is a real number and n is an odd natural number. The end behavior of the graph of $y=f(x)$ matches one of the following:

- For $a>0$, as $x\to -\infty$, $f(x)\to -\infty$ and as $x\to\infty$, $f(x)\to\infty$
- For $a<0$, as $x\to -\infty$, $f(x)\to\infty$ and as $x\to\infty$, $f(x)\to -\infty$

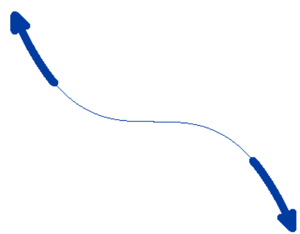
Graphically:

Figure 2.2. 13



$a>0$

Figure 2.2. 14



$a<0$

The following allows us to move beyond the end behavior of simple monomials and apply our results to all polynomial functions.

End Behavior for Polynomial Functions

The end behavior of a polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$, with degree $n>0$, matches the end behavior of $y=a_nx^n$.

To see why the end behavior of a polynomial matches the end behavior of its leading term, let us look at the specific example $f(x)=4x^3-x+5$. To examine the end behavior, we look at the behavior of f and the behavior of its leading term, $4x^3$, as $x\to\pm\infty$. The following table shows values of $4x^3$ and values of $f(x)$ for various input values of x.

x	$4x^3$	$f(x)=4x^3-x+5$
-1000	$-4,000,000,000$	$-3,999,998,995$
-100	$-4,000,000$	$-3,999.895$
-10	$-4,000$	$-3,985$
10	$4,000$	$3,995$
100	$4,000,000$	$3,999,905$
1000	$4,000,000,000$	$3,999,999,005$

As $x \to \pm\infty$, the table shows us that $f(x) \approx 4x^3$. Our next example shows how end behavior and multiplicity allow us to sketch a decent graph without calculating function values.

Example 2.2.5. Sketch a rough graph of $f(x)=x^3(2-x)(x+3)^2$ using end behavior and multiplicities of zeros.

Solution. The end behavior of the graph of f will match that of its leading term. To find the leading term, we multiply the leading terms of each factor to get $x^3(-x)(x)^2=-x^6$. Thus, the end behavior will be the same as that of $-x^6$. This tells us that the graph of $y=f(x)$ starts and ends below the *x*-axis. In other words, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

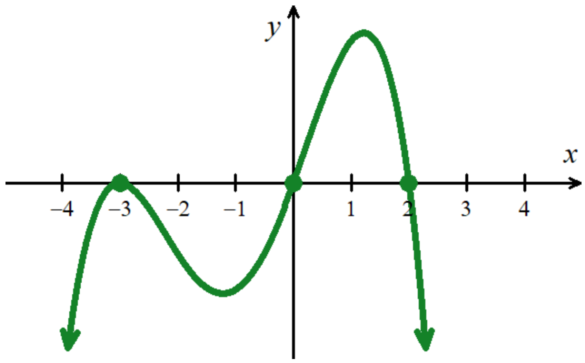
Next, we find the zeros of $f(x)=x^3(2-x)(x+3)^2$. Fortunately, f is factored.[9] Setting each factor equal to 0 gives us $x=0$, $x=2$, and $x=-3$ as zeros. To determine multiplicities, we look at the exponent on each factor to identify the number of occurrences of that factor. Since the zero $x=0$ is a result of the factor x^3, its multiplicity is 3; $x=2$ results from $(2-x)^1$ so has a multiplicity of 1; $x=-3$ comes from the factor $(x+3)^2$ for a multiplicity of 2. Thus,

- $x=0$ is a zero of odd multiplicity 3, so the graph will cross through the *x*-axis at $(0,0)$.
- $x=2$ is a zero of odd multiplicity 1, so the graph will cross through the *x*-axis at $(2,0)$.
- $x=-3$ is a zero of even multiplicity 2, so the graph will only touch the *x*-axis at $(-3,0)$.

Putting all of this information together, we sketch a smooth, continuous, curve that includes the *x*-intercepts and the intervals above/below the *x*-axis. Following is a rough graph of $f(x)=x^3(2-x)(x+3)^2$.

[9] Obtaining the factored form of a polynomial is the main focus of the next few sections.

Figure 2.2. 15

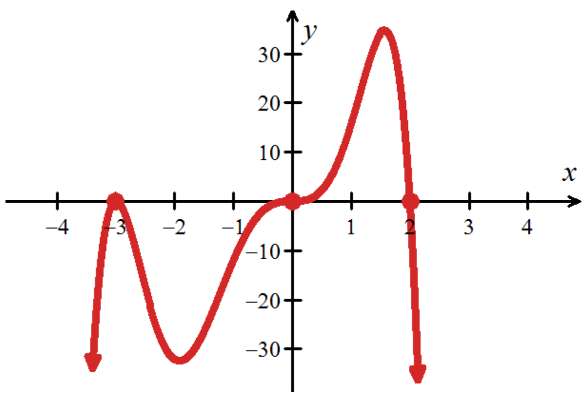


A rough graph of $y = f(x)$

□

In Calculus, you will learn techniques for drawing a more precise graph. For now, to make the graph more precise, simply plot additional points. Below is a precise graph of $y = f(x)$, to compare with the rough graph shown above. The graph's behavior may be verified by plotting points corresponding to x-values such as -3.5, -2, -1, -0.5, 0.5, 1, 1.5, and 2.5.

Figure 2.2. 16



Precise graph of $y = f(x)$

A general strategy for graphing polynomial functions follows.

To Graph a Polynomial Function:

1. Find the real zeros of the function and their multiplicities.
2. Determine the behavior of the graph at each *x*-intercept. (Does the graph cross the *x*-axis at the intercept?)
3. Find the *y*-intercept of the graph of the function.
4. Determine the end behavior of the graph of the function.
5. Sketch a smooth curve having all of the above properties.

For a more accurate graph, plot additional points.

Applications

We end this section with an example that shows how polynomials of higher degree arise 'naturally' in even the most basic geometric applications.

Example 2.2.6. A box with no top is to be fashioned from a 10 inch by 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square that is removed from each corner.

Figure 2.2. 17

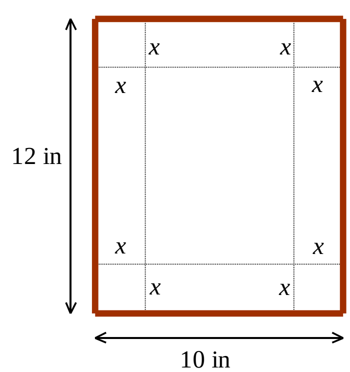
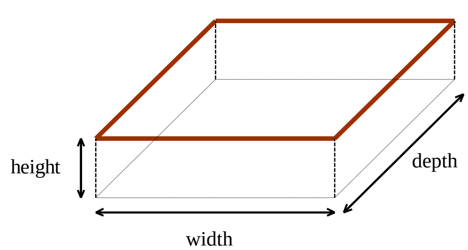


Figure 2.2. 18



1. Find the volume V of the box as a function of x. Include an appropriate applied domain.
2. Use graphing technology to graph $y = V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

Solution.

1. From Geometry, we know that volume = width × height × depth. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from

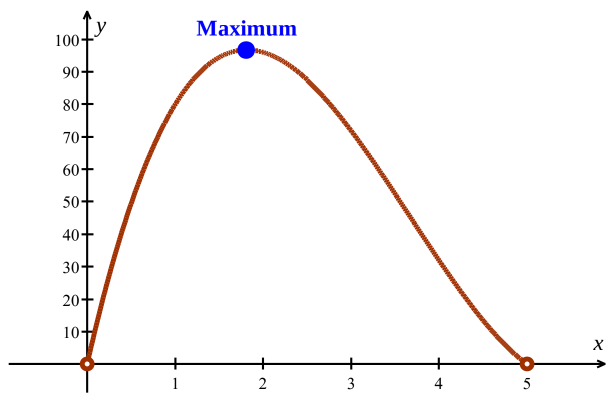
each corner leaves $10-2x$ inches for the width.[10] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12-2x$ inches remaining. As a function of *x*, the volume is[11]

$$\begin{aligned} V(x) &= x(10-2x)(12-2x) \\ &= x\left(120-44x+4x^2\right) \\ &= 120x-44x^2+4x^3 \end{aligned}$$

To find a suitable domain, we note that to make a box at all we need $x>0$. Also, the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10-2x>0$ or $x<5$. Hence, our volume is $V(x)=4x^3-44x^2+120x$ and the applied domain is $0<x<5$.

2. Using graphing technology, such as a graphing calculator or online graphing tool[12], we see that the graph of $y=V(x)$, $0<x<5$, has a maximum value.

Figure 2.2. 19



Using graphing technology, we find the maximum point occurs when $x \approx 1.81$ and $y \approx 96.77$. This yields a height of $x \approx 1.81$ inches, a width of $10-2x \approx 6.38$ inches, and a depth of $12-2x \approx 8.38$ inches. The *y*-coordinate gives us the maximum volume, which is approximately 96.77 cubic inches.

□

[10] There is no harm in taking an extra step here and making sure this makes sense. If we removed a 1×1 inch square from each corner, then the width would be 8 inches, so removing a square with sides of length *x* inches would leave 10-2*x* inches.
[11] When we write *V* (*x*), it is in the context of function notation, not the volume *V* times the quantity *x*.
[12] Wolfram Alpha, Desmos, or GeoGebra are possiblilities. Many free online graphing calculators are available as well.

2.2 Exercises

1. If a polynomial function is in factored form, what first step would be useful to determine the degree of the function?

2. In general, explain the end behavior of the graph of a polynomial function with odd degree if the leading coefficient is positive.

In Exercises 3 – 12, find the degree, leading term, leading coefficient, constant term, and end behavior of the given polynomial.

3. $f(x) = 4 - x - 3x^2$
4. $g(x) = 3x^5 - 2x^2 + x + 1$
5. $q(r) = 1 - 16r^4$
6. $Z(b) = 42b - b^3$
7. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
8. $s(t) = -4.9t^2 + v_0 t + s_0$
9. $P(x) = (x-1)(x-2)(x-3)(x-4)$
10. $p(t) = -t^2(3-5t)(t^2+t+4)$
11. $f(x) = -2x^3(x+1)(x+2)^2$
12. $G(t) = 4(t-2)^2\left(t + \frac{1}{2}\right)$

In Exercises 13 – 22, find the *x*-intercept(s) of the given polynomial, the multiplicities of the corresponding zeros, and the *y*-intercept. Use this information, along with end behavior where necessary, to sketch a rough graph of the polynomial.

13. $a(x) = x(x+2)^2$
14. $g(x) = x(x+2)^3$
15. $f(x) = -2(x-2)^2(x+1)$
16. $g(x) = (2x+1)^2(x-3)$
17. $F(x) = x^3(x+2)^2$
18. $P(x) = (x-1)(x-2)(x-3)(x-4)$
19. $Q(x) = (x+5)^2(x-3)^4$
20. $h(x) = x^2(x-2)^2(x+2)^2$
21. $H(t) = (3-t)(t+1)^2$
22. $Z(b) = b(49-b^2)$

In Exercises 23 – 27, create a polynomial p that has the desired characteristics. You may leave the polynomial in factored form.

23. ▪ The *x*-intercepts of p are $(\pm1, 0)$ and $(\pm2, 0)$.
    ▪ The leading term of $p(x)$ is $117x^4$.

24. ▪ The zeros of p are $x=1$ and $x=3$.
   ▪ $x=3$ is a zero of multiplicity 2.
   ▪ $p(x)$ has the y-intercept $(0,45)$.

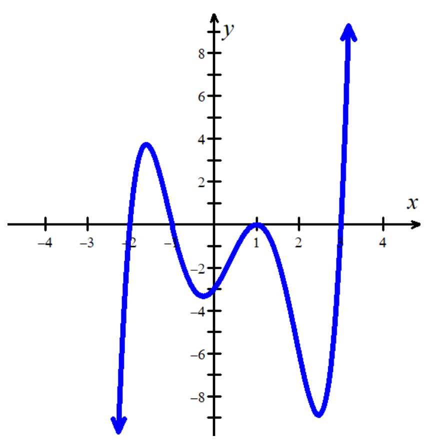
25. ▪ The solutions to $p(x)=0$ are $x=\pm 3$ and $x=6$.
   ▪ The leading term of $p(x)$ is $7x^4$.
   ▪ The point $(-3,0)$ is a local minimum on the graph of $y=p(x)$.

26. ▪ The solutions to $p(x)=0$ are $x=\pm 3$, $x=-2$ and $x=4$.
   ▪ The leading term of $p(x)$ is $-x^5$.
   ▪ The point $(-2,0)$ is a local maximum on the graph of $y=p(x)$.

27. ▪ p is of degree 4.
   ▪ As $x\to\infty$, $p(x)\to-\infty$.
   ▪ p has exactly three x-intercepts: $(-6,0)$, $(1,0)$ and $(117,0)$.
   ▪ The graph of $y=p(x)$ touches the x-axis at $(1,0)$.

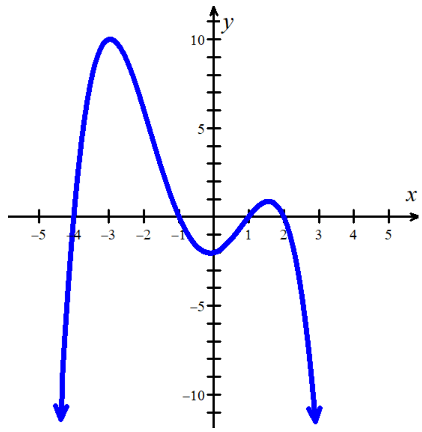
28. Write an equation for the polynomial $p(x)$, of degree 5, that is graphed below. Leave your equation in factored form.

Figure Ex2.2. 1



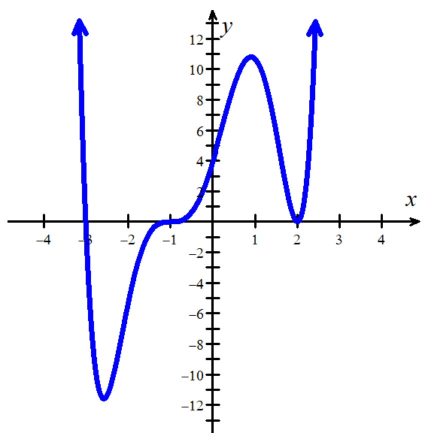
29. Write an equation for the polynomial $p(x)$, of degree 4, that is graphed below. Leave your equation in factored form.

Figure Ex2.2. 2



30. Write an equation for the polynomial $p(x)$, of degree 6, that is graphed below. Note that the zero $x=-1$ has multiplicity 3. Leave your equation in factored form.

Figure Ex2.2. 3



In Exercises 31 – 36, given the pair of functions f and g, sketch the graph of $y=g(x)$ by starting with the graph of $y=f(x)$ and applying transformations. Track at least three points of your choice through the transformations. State the domain and range of g.

31. $f(x)=x^3$, $g(x)=(x+2)^3+1$
32. $f(x)=x^4$, $g(x)=(x+2)^4+1$
33. $f(x)=x^4$, $g(x)=2-3(x-1)^4$
34. $f(x)=x^5$, $g(x)=-x^5-3$
35. $f(x)=x^5$, $g(x)=(x+1)^5+10$
36. $f(x)=x^6$, $g(x)=8-x^6$

37. Use the Intermediate Value Theorem to prove that $f(x)=x^3-9x+5$ has a real zero in each of the following intervals: $(-4,-3)$, $(0,1)$ and $(2,3)$.

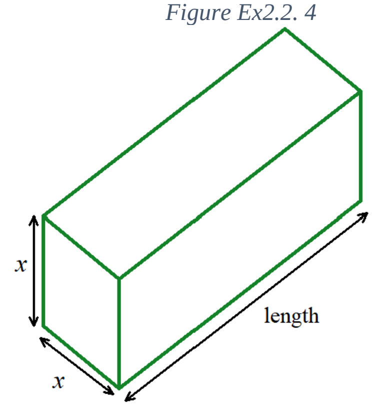
38. Use the Intermediate Value Theorem to confirm that $f(x)=x^3-100x+2$ has at least one real zero between $x=0.01$ and $x=0.1$.

39. Show that the function $f(x)=x^3-5x^2+3x+6$ has at least two real zeros between $x=1$ and $x=4$.

40. Rework **Example 2.2.6**, assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Then, using scissors and tape, construct the box. Are you surprised?

41. According to US Postal regulations, a rectangular shipping box must satisfy the inequality $\text{Length}+\text{Girth}\le 130\text{ inches}$ for Parcel Post and $\text{Length}+\text{Girth}\le 108\text{ inches}$ when using a private company's shipping services. Let's assume we have a closed rectangular box with a square face of side length x, as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so, in our case, it is the sum of the four sides of the square, $4x$.

Figure Ex2.2. 4



(a) Assuming that we'll be mailing a box via Parcel Post where $\text{Length}+\text{Girth}=130\text{ inches}$, express the length of the box in terms of x and then express the volume V of the box in terms of x.

(b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

(c) Repeat parts (a) and (b) if the box is shipped using a private company's shipping services.

42. Suppose the profit P, in thousands of dollars, from producing and selling x hundred LCD TV's is given by $P(x)=-5x^3+35x^2-45x-25$ for $0\le x\le 10.07$. Use graphing technology to graph $y=P(x)$ and determine the number of TV's that should be sold to maximize profit. What is the maximum profit?

43. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy revised their profit function and now use $P(x) = -0.03x^3 + 3x^2 + 25x - 250$, for $x \geq 0$. Use graphing technology to find the production level x that maximizes the profit made by producing and selling x PortaBoy game systems.

44. Show that the end behavior of a linear function $f(x) = mx + b$, $m \neq 0$, is as it should be according to the results we've established in this section for polynomials of odd degree. (That is, show that the graph of a linear function is 'up on one side' and 'down on the other' just like the graph of $y = a_n x^n$ for odd numbers n.)

45. Here are a few questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree n have? How few local extrema?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an *x*-intercept yield a local extremum? Can it yield an absolute extremum?

(h) If the *y*-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

2.3 Using Synthetic Division to Factor Polynomials

Learning Objectives

- Use division to factor polynomials and determine zeros.
- Use synthetic division to simplify the division process.
- Use the Remainder Theorem to find function values of polynomials.
- Use the Factor Theorem to relate zeros to factors of polynomials.

Using Division to Find Zeros of Polynomials

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned in Intermediate Algebra, this equation foils[13] us at every turn. Should we happen to guess (correctly) that $x = 2$ is a zero, there must be a factor of $(x-2)$ lurking around in the factorization of $f(x)$. How could we use the factor $(x-2)$ to find this factorization? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$\begin{array}{r} x^2 + 6x + 7 \\ x-2\overline{\smash{\big)}\,x^3 + 4x^2 - 5x - 14} \\ -\underline{\left(x^3 - 2x^2\right)} \qquad\qquad \\ 6x^2 - 5x \qquad\;\; \\ -\underline{\left(6x^2 - 12x\right)} \qquad\;\; \\ 7x - 14 \\ -\underline{\left(7x - 14\right)} \\ 0 \end{array}$$

This means $\left(x^3 + 4x^2 - 5x - 14\right) \div (x-2) = x^2 + 6x + 7$ or, after multiplying both sides by $(x-2)$,

$x^3 + 4x^2 - 5x - 14 = \left(x^2 + 6x + 7\right)(x-2)$.

[13] Pun intended.

To find the zeros of $f(x)=x^3+4x^2-5x-14$, we now solve $\left(x^2+6x+7\right)(x-2)=0$. We get $x^2+6x+7=0$ or $x-2=0$. Since x^2+6x+7 does not factor nicely, we apply the Quadratic Formula to get $x=-3\pm\sqrt{2}$. From $x-2=0$, we have our 'known' zero of $x=2$.

The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

> **Theorem 2.2. Polynomial Division:** Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x)=d(x)q(x)+r(x)$, where either $r(x)=0$ or the degree of r is strictly less than the degree of d.

We can rewrite $p(x)=d(x)q(x)+r(x)$ as $\dfrac{p(x)}{d(x)}=q(x)+\dfrac{r(x)}{d(x)}$ or, in long division format, as

$$\begin{array}{r} q(x) \\ d(x)\overline{)\,p(x)} \\ \vdots \\ \hline r(x) \end{array}$$

In either of these formats, you may recall that the polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x)=0$ then d is called a **factor** of p. The proof of **Theorem 2.2** is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts that are the basis for the rest of the chapter.

The Remainder and Factor Theorems

> **Theorem 2.3. The Remainder Theorem:** Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x-c$, the remainder is $p(c)$.

The proof of **Theorem 2.3** is a direct consequence of **Theorem 2.2**. When a polynomial is divided by $x-c$, the remainder is either zero or has degree less than the degree of $x-c$. Since $x-c$ is degree 1, the degree of the remainder must be zero, which means the remainder is a constant. Hence, in either case, $p(x)=(x-c)q(x)+r$ where r, the remainder, is a real number, possibly zero. We have the following.

$$
\begin{aligned}
p(c) &= (c-c)q(c)+r \\
&= 0 \cdot q(c)+r \\
&= r
\end{aligned}
$$

So we get $r = p(c)$ as required. We have one more result to present.

Theorem 2.4. The Factor Theorem: Suppose p is a polynomial. The real number c is a zero of p if and only if $(x-c)$ is a factor of $p(x)$.

The proof of the Factor Theorem is a consequence of what we already know. If $(x-c)$ is a factor of $p(x)$, this means $p(x)=(x-c)q(x)$ for some polynomial q. Hence, $p(c)=(c-c)q(x)=0$, so c is a zero of p.

Conversely, if c is a zero of p, then $p(c)=0$. In this case, the Remainder Theorem tells us the remainder when $p(x)$ is divided by $x-c$, namely $p(c)$, is zero, which means that $(x-c)$ is a factor of p. What we have established is the fundamental connection between zeros of polynomials and factors of polynomials.

Synthetic Division – Why it Works

We can find a more efficient way to divide polynomials by quantities of the form $x-c$. Let us take a closer look at the long division we performed at the beginning of this section. Below is that division with the minus signs in front of parentheses distibuted.

$$
\begin{array}{r|l}
 & x^2+6x+7 \\
\hline
x-2 & x^3+4x^2-5x-14 \\
 & \underline{-x^3+2x^2} \\
 & \quad 6x^2-5x \\
 & \quad \underline{-6x^2+12x} \\
 & \qquad 7x-14 \\
 & \qquad \underline{-7x+14} \\
 & \qquad\qquad 0
\end{array}
$$

Observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposites of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely $-5x$ and -14) are not really necessary to recopy, so we omit them, too.

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\overline{\big)\,x^3+4x^2-5x-14} \\
\underline{2x^2} \\
6x^2 \\
\underline{12x} \\
7x \\
\underline{14} \\
0
\end{array}
$$

Now let's 'move things up' a bit, for reasons that will become clear in a moment, and copy the x^3 into the last row.

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\overline{\big)\,x^3+4x^2-5x-14} \\
\underline{2x^2\quad 12x\quad 14} \\
x^3\quad 6x^2\quad 7x\quad 0
\end{array}
$$

Note that each term in the last row is now obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r|rrrr}
-2 & x^3 & +4x^2 & -5x & -14 \\
 & & 2x^2 & 12x & 14 \\
\hline
 & x^3 & 6x^2 & 7x & 0
\end{array}
$$

To streamline things further, recall that the $2x^2$, $12x$, and 14 in the second row came from multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, first by -2 and then by -1. The result is multiplication by 2, so we replace the -2 in the divisor with 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms.

$$
\begin{array}{r|rrrr}
2 & 1 & 4 & -5 & -14 \\
 & & 2 & 12 & 14 \\
\hline
 & 1 & 6 & 7 & 0
\end{array}
$$

Synthetic Division – How it Works

We have constructed a **synthetic division tableau**. Let us rework our division problem using this tableau. To divide $x^3+4x^2-5x-14$ by $x-2$, we write $-(-2)=2$ in the place of the divisor,[14] and the coefficients and constant term of $x^3+4x^2-5x-14$ in the place of the dividend. Note that if the dividend was not already in descending order, we would first write it in descending order. Furthermore, if a term is missing, we place a zero in its position. We then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, we take the 2 from the divisor and multiply by the 1 that was brought down, resulting in 2. We write the 2 underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now we take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, we take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \mathbf{0} \end{array}$$

The first three numbers in the last row are the coefficients of the quotient polynomial. Having divided a third degree polynomial by a first degree polynomial, the quotient is the second degree polynomial x^2+6x+7. The remainder is 0.

Synthetic division is a time saver, but only works for divisors of the form $x-c$. Note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree.

[14] Note that 2 is the zero of the divisor $x-2$.

Example 2.3.1. Use synthetic division to perform the following polynomial divisions. Find the quotient and the remainder polynomials, then write the dividend, quotient, and remainder in the form $p(x)=d(x)q(x)+r(x)$, as given in **Theorem 2.2**.

1. $(5x^3-2x^2+1)\div(x-3)$

2. $(x^3+8)\div(x+2)$

Solution.

1. For $(5x^3-2x^2+1)\div(x-3)$, when setting up the synthetic division tableau, we write $-(-3)=3$ in the place of the divisor. We enter 0 at the location of the coefficient of x since there is no x term in the dividend. Doing so, we have the following:

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & 118 \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13, and 39. Our quotient is $q(x)=5x^2+13x+39$ and the remainder is $r(x)=118$. Thus, we have $5x^3-2x^2+1=(x-3)(5x^2+13x+39)+118$.

2. In dividing x^3+8 by $x+2$, we use $-(2)=-2$ for the divisor and 0 at the locations of the missing terms, x^2 and x, then proceed as before.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

We get the quotient $q(x)=x^2-2x+4$ and the remainder $r(x)=0$. This gives us $x^3+8=(x+2)(x^2-2x+4)$.

□

Our next example demonstrates using synthetic division with a divisor of the form $ax+b$ where $a\neq 1$.

Example 2.3.2. Use synthetic division to perform the polynomial division $\dfrac{4-8x-12x^2}{2x-3}$. Then write the dividend, quotient and remainder in the form $\dfrac{p(x)}{d(x)}=q(x)+\dfrac{r(x)}{d(x)}$.

Solution. To divide $4-8x-12x^2$ by $2x-3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2-8x+4$. Second, since synthetic division is designed only for factors of the form $x-c$, we factor $2x-3$ as $2\left(x-\frac{3}{2}\right)$. Our strategy is to first divide $-12x^2-8x+4$ by 2 to get $-6x^2-4x+2$. Next, we divide by $\left(x-\frac{3}{2}\right)$. The tableau becomes

$$\begin{array}{c|ccc} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & -\frac{35}{2} \end{array}$$

We get $-6x^2-4x+2=\left(x-\frac{3}{2}\right)(-6x-13)-\frac{35}{2}$. Multiplying both sides by 2 and distributing gives $-12x^2-8x+4=(2x-3)(-6x-13)-35$, for a final answer of $\dfrac{4-8x-12x^2}{2x-3}=-6x-13-\dfrac{35}{2x-3}$.

□

The next example pulls together all of the concepts discussed in this section.

Example 2.3.3. Let $p(x)=2x^3-5x+3$.

1. Find $p(-2)$ using the Remainder Theorem. Check your answer by substitution.
2. Use the fact that $x=1$ is a zero of p to find all real zeros of p.

Solution.

1. The Remainder Theorem states that $p(c)$ is the remainder when $p(x)$ is divided by $x-c$. Here, $c=-2$, and so $p(-2)$ is the remainder when $p(x)$ is divided by $x-(-2)=x+2$. We set up our synthetic division tableau below. We are careful to write 0 in the location of the missing x^2 term.

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & -3 \end{array}$$

According to the Remainder Theorem, $p(-2)=-3$. We can check this by direct substitution into the formula $p(x)$.

$$\begin{aligned} p(-2) &= 2(-2)^3-5(-2)+3 \\ &= -16+10+3 \\ &= -3 \end{aligned}$$

2. The Factor Theorem tells us that since $x=1$ is a zero of p, $(x-1)$ is a factor of $p(x)$. To factor $p(x)$, we divide as follows.

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & 0 \end{array}$$

We get a remainder of zero, which verifies that $p(1)=0$. Our quotient polynomial is $q(x)=2x^2+2x-3$, from which we find $p(x)=(x-1)(2x^2+2x-3)$. To find the remaining zeros of p, we need to solve $2x^2+2x-3=0$ for x. Since this doesn't factor nicely, we use the Quadratic Formula:

$$\begin{aligned} x &= \frac{-2\pm\sqrt{(2)^2-4(2)(-3)}}{2(2)} \\ &= \frac{-2\pm\sqrt{28}}{4} \\ &= \frac{-2\pm 2\sqrt{7}}{4} \\ &= \frac{-1\pm\sqrt{7}}{2} \end{aligned}$$

Thus, the real zeros of p are $x=1$, $x=\frac{-1+\sqrt{7}}{2}$, and $x=\frac{-1-\sqrt{7}}{2}$.

□

In **Section 2.2**, we introduced the multiplicity of a zero. The role of multiplicity in finding zeros of a polynomial is illustrated in the next example.

Example 2.3.4. Let $p(x)=4x^4-4x^3-11x^2+12x-3$. Given that $x=\frac{1}{2}$ is a zero of multiplicity 2, find all real zeros of p.

Solution. We set up for synthetic division. Since we are told the multiplicity of $\frac{1}{2}$ is two, after the first division by $x-\frac{1}{2}$, we continue our tableau and divide the quotient by $x-\frac{1}{2}$ again.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\ & \downarrow & 2 & -1 & -6 & 3 \\ \hline \frac{1}{2} & 4 & -2 & -12 & 6 & 0 \\ & \downarrow & 2 & 0 & -6 & \\ \hline & 4 & 0 & -12 & 0 & \end{array}$$

From the first division, we have

$$4x^4-4x^3-11x^2+12x-3=\left(x-\frac{1}{2}\right)\left(4x^3-2x^2-12x+6\right)$$

The second division gives us

$$4x^3-2x^2-12x+6=\left(x-\frac{1}{2}\right)\left(4x^2-12\right)$$

We combine the results of the first and second division to get

$$4x^4-4x^3-11x^2+12x-3=\left(x-\frac{1}{2}\right)^2\left(4x^2-12\right)$$

To find the remaining zeros of p, we set $4x^2-12=0$ and get

$$\begin{aligned}4x^2-12&=0\\ x^2&=3\\ x&=\pm\sqrt{3}\end{aligned}$$

□

In the previous example, we found $x=\sqrt{3}$ and $x=-\sqrt{3}$ are zeros of p, from which the Factor Theorem guarantees that both $\left(x-\sqrt{3}\right)$ and $\left(x-\left(-\sqrt{3}\right)\right)=\left(x+\sqrt{3}\right)$ are factors of p, as demonstrated below.

$$\begin{aligned}4x^4-4x^3-11x^2+12x-3&=\left(x-\frac{1}{2}\right)^2\left(4x^2-12\right)\\ &=\left(x-\frac{1}{2}\right)^2(4)\left(x^2-3\right)\\ &=(4)\left(x-\frac{1}{2}\right)^2\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)\end{aligned}$$

The next section provides tools to help us identify real numbers that may be zeros. We close this section with the summary of a connection between several concepts that have been presented.

Connections Between Zeros, Factors, and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n\geq 1$. The following statements are equivalent.

- The real number c is a zero of p.
- $p(c)=0$
- $x=c$ is a solution to the polynomial equation $p(x)=0$.
- $(x-c)$ is a factor of $p(x)$.
- The point $(c,0)$ is an x-intercept of the graph of $y=p(x)$.

2.3 Exercises

1. If division of a polynomial (of degree at least 1) by $x+4$ results in a remainder of zero, what can we conclude?
2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

In Exercises 3 – 8, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.

3. $\left(4x^2+3x-1\right)\div(x-3)$
4. $\left(2x^3-x+1\right)\div\left(x^2+x+1\right)$
5. $\left(5x^4-3x^3+2x^2-1\right)\div\left(x^2+4\right)$
6. $\left(-x^5+7x^3-x\right)\div\left(x^3-x^2+1\right)$
7. $\left(9x^3+5\right)\div(2x-3)$
8. $\left(4x^2-x-23\right)\div\left(x^2-1\right)$

In Exercises 9 – 22, use synthetic division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.

9. $\left(3x^2-2x+1\right)\div(x-1)$
10. $\left(x^2-5\right)\div(x-5)$
11. $\left(3-4x-2x^2\right)\div(x+1)$
12. $\left(4x^2-5x+3\right)\div(x+3)$
13. $\left(x^3+8\right)\div(x+2)$
14. $\left(4x^3+2x-3\right)\div(x-3)$
15. $\left(18x^2-15x-25\right)\div\left(x-\frac{5}{3}\right)$
16. $\left(4x^2-1\right)\div\left(x-\frac{1}{2}\right)$
17. $\left(2x^3+x^2+2x+1\right)\div\left(x+\frac{1}{2}\right)$
18. $\left(3x^3-x+4\right)\div\left(x-\frac{2}{3}\right)$
19. $\left(2x^3-3x+1\right)\div\left(x-\frac{1}{2}\right)$
20. $\left(4x^4-12x^3+13x^2-12x+9\right)\div\left(x-\frac{3}{2}\right)$
21. $\left(x^4-6x^2+9\right)\div\left(x-\sqrt{3}\right)$
22. $\left(x^6-6x^4+12x^2-8\right)\div\left(x+\sqrt{2}\right)$

In Exercises 23 – 32, determine $p(c)$ using the Remainder Theorem for the given polynomial function and value of c. If $p(c)=0$, write the polynomial in the factored form $p(x)=(x-c)q(x)$.

23. $p(x)=2x^2-x+1$, $c=4$
24. $p(x)=4x^2-33x-180$, $c=12$

25. $p(x)=2x^3-x+6$, $c=-3$

26. $p(x)=x^3+2x^2+3x+4$, $c=-1$

27. $p(x)=3x^3-6x^2+4x-8$, $c=2$

28. $p(x)=8x^3+12x^2+6x+1$, $c=-\frac{1}{2}$

29. $p(x)=x^4-2x^2+4$, $c=\frac{3}{2}$

30. $p(x)=6x^4-x^2+2$, $c=-\frac{2}{3}$

31. $p(x)=x^4+x^3-6x^2-7x-7$, $c=-\sqrt{7}$

32. $p(x)=x^2-4x+1$, $c=2-\sqrt{3}$

In Exercises 33 – 44, you are given a polynomial equation along with one or more of its zeros, one of its factors, or its graph. Use the techniques of this section to find the remaining real zeros and factor the polynomial.

33. $p(x)=x^3-6x^2+11x-6$, zero $x=1$

34. $p(x)=x^3-24x^2+192x-512$, factor $x-8$

35. $p(x)=2x^3-3x^2-11x+6$, zero $x=\frac{1}{2}$

36. $p(x)=2x^3-x^2-10x+5$, factor $x-\frac{1}{2}$

37. $p(x)=3x^4+10x^3+7x^2-4x-4$, zeros $x=\frac{2}{3}$ and $x=-2$

38. $p(x)=x^4-2x^3-11x^2+6x+24$, zeros $x=-2$ and $x=4$

39. $p(x)=4x^4-28x^3+61x^2-42x+9$, zero $x=\frac{1}{2}$ of multiplicity 2

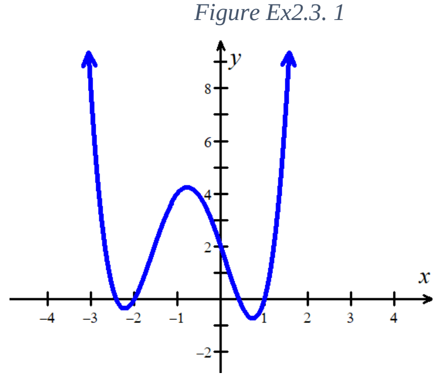
40. $p(x)=x^5+2x^4-12x^3-38x^2-37x-12$, zero $x=-1$ of multiplicity 3

41. $p(x)=125x^5-275x^4-2265x^3-3213x^2-1728x-324$, zero $x=-\frac{3}{5}$ of multiplicity 3

42. $p(x)=x^2-2x-2$, zero $x=1-\sqrt{3}$

43. $p(x)=x^4+3x^3-x^2-5x+2$

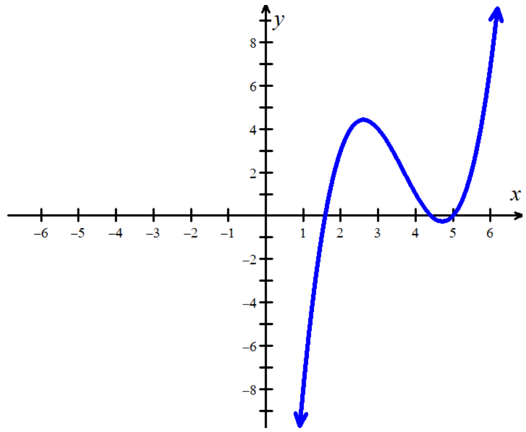
Figure Ex2.3. 1



Graph of $p(x)=x^4+3x^3-x^2-5x+2$

44. $p(x) = x^3 - 11x^2 + 37x - 35$

Figure Ex2.3. 2



Graph of $p(x) = x^3 - 11x^2 + 37x - 35$

45. Find a quadratic polynomial with integer coefficients that has $x = \frac{3}{5} \pm \frac{\sqrt{29}}{5}$ as its real zeros.

2.4 Real Zeros of Polynomials

Learning Objectives

- Find possible (potential) rational zeros using the Rational Zeros Theorem.
- Find real zeros of a polynomial and their multiplicities.

In **Section 2.3**, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results that will help us determine good candidates to test using synthetic division.

Finding Potential Rational Zeros

The following theorem gives us a list of possible real zeros.

Theorem 2.5. Rational Zeros Theorem: Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a polynomial of degree n with $n\geq 1$, and a_0, a_1, $\ldots$, a_n are integers. If r is a rational zero of f, then r is of the form $\frac{p}{q}$, where the integer p is a factor of the constant term a_0 and the integer q is a factor of the leading coefficient a_n.

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in our list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. We will not offer a proof here but will note that you have been using this idea in factoring quadratic expressions.

Example 2.4.1. Let $f(x)=2x^3-3x^2-8x-3$. Use the Rational Zeros Theorem to list all possible rational zeros of f.

Solution. To generate a complete list of possible rational zeros, we need to take each of the factors of the constant term, $a_0=-3$, and divide them by each of the factors of the leading coefficient, $a_3=2$. The factors of -3 are ±1 and ±3. The factors of 2 are ±1 and ±2, so the Rational Zeros Theorem gives the list $\frac{\pm1}{\pm1}$, $\frac{\pm1}{\pm2}$, $\frac{\pm3}{\pm1}$, and $\frac{\pm3}{\pm2}$, which is the same as ±1, $\pm\frac{1}{2}$, ±3, and $\pm\frac{3}{2}$.

□

The next example pulls together the Rational Zeros Theorem and the use of synthetic division, from **Section 2.3**, to determine which potential rational zeros, if any, are zeros of the polynomial.

Example 2.4.2. Find all of the rational zeros of $f(x)=2x^3-3x^2-8x-3$.

Solution. From **Example 2.4.1**, we have potential rational zeros of ± 1, $\pm\frac{1}{2}$, ± 3, and $\pm\frac{3}{2}$. We first try our potential zero of 1.

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -8 & -3 \\ & \downarrow & 2 & -1 & -9 \\ \hline & 2 & -1 & -9 & -12 \end{array}$$

Since the remainder is not zero, we know that $x=1$ is not a zero. We continue to the next possible zero of -1.

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -8 & -3 \\ & \downarrow & -2 & 5 & 3 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

The remainder of 0 tells us that $x=-1$ is a zero. We have the additional information that $(x+1)$ is a factor and that $f(x)=(x+1)(2x^2-5x-3)$. While we could continue with synthetic division, once we have a second degree factor it is easier to find its zeros directly. So, we now solve $2x^2-5x-3=0$. Factoring gives us $(2x+1)(x-3)=0$, from which we get the additional zeros of $x=-\frac{1}{2}$ and $x=3$. Notice that these were also listed as potential rational zeros. The zeros of $f(x)$ are -1, $-\frac{1}{2}$, and 3.

□

With the next example, we revisit multiplicities, while relying on synthetic division and the Rational Zeros Theorem to factor polynomials to the point where we can identify non-rational zeros.

Example 2.4.3. Find the real zeros of $f(x)=x^4-2x^3-7x^2+16x-8$ and their multiplicities.

Solution. We begin by determining the potential rational zeros of f. The factors of the constant term, $a_0=-8$, are ± 1, ± 2, ± 4, and ± 8. Dividing the factors of a_0 by the factors of $a_4=1$, which include only ± 1, we have potential rational zeros of ± 1, ± 2, ± 4, and ± 8.

We begin by testing the potential positive zero of 1.

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & -7 & 16 & -8 \\ & \downarrow & 1 & -1 & -8 & 8 \\ \hline & 1 & -1 & -8 & 8 & 0 \end{array}$$

We see that $x=1$ is a zero. By continuing the tableau with 1 again, we discover that it is a zero of multiplicity at least two.

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & -7 & 16 & -8 \\ & \downarrow & 1 & -1 & -8 & 8 \\ \hline 1 & 1 & -1 & -8 & 8 & 0 \\ & \downarrow & 1 & 0 & -8 & \\ \hline & 1 & 0 & -8 & 0 & \end{array}$$

Synthetic division additionally provides us with a factored version of f : $f(x)=(x-1)^2(x^2-8)$.

Setting $x^2-8=0$, we find $x=\pm\sqrt{8}$, or $x=\pm 2\sqrt{2}$. Thus, $x=2\sqrt{2}$ and $x=-2\sqrt{2}$ are two zeros, each of multiplicity one, and $x=1$ is a zero of multiplicity two.

□

In the previous example, having found all of its zeros, we may use the Factor Theorem to write $f(x)$ as $f(x)=(x-1)^2\left(x-2\sqrt{2}\right)\left(x-\left(-2\sqrt{2}\right)\right)$. Our next example reminds us of the role finding zeros plays in solving equations.

Example 2.4.4. Find the real solutions to the equation $2x^5+6x^3+3=3x^4+8x^2$.

Solution. Finding the real solutions to $2x^5+6x^3+3=3x^4+8x^2$ is the same as finding the real solutions to $2x^5-3x^4+6x^3-8x^2+3=0$. In other words, we are looking for the real zeros of $p(x)=2x^5-3x^4+6x^3-8x^2+3$. The constant term, $a_0=3$, has factors ± 1 and ± 3 while $a_5=2$ has factors ± 1 and ± 2. We find that the potential rational zeros are ± 1, $\pm\frac{1}{2}$, ± 3, and $\pm\frac{3}{2}$. Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr} 1 & 2 & -3 & 6 & -8 & 0 & 3 \\ & \downarrow & 2 & -1 & 5 & -3 & -3 \\ \hline 1 & 2 & -1 & 5 & -3 & -3 & 0 \\ & \downarrow & 2 & 1 & 6 & 3 & \\ \hline & 2 & 1 & 6 & 3 & 0 & \end{array}$$

Thus, $x=1$ is a zero of multiplicity at least 2 and we have $p(x)=(x-1)^2(2x^3+x^2+6x+3)$. To find remaining zeros, we set $2x^3+x^2+6x+3=0$ and return to synthetic division.[15]

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 1 & 6 & 3 \\ & \downarrow & -1 & 0 & -3 \\ \hline & 2 & 0 & 6 & 0 \end{array}$$

We have an additional zero of $x=-\frac{1}{2}$ and the factorization $2x^3+x^2+6x+3=\left(x+\frac{1}{2}\right)(2x^2+6)$. Since $2x^2+6=0$ has no real solutions, our solutions to the original equation are only $x=1$ and $x=-\frac{1}{2}$.

□

Note that we could have checked $x=-\frac{1}{2}$ in the original polynomial. However, to save time we have used the quotient polynomial.

[15] Factoring by grouping could also be used here.

2.4 Exercises

1. Explain why the Rational Zeros Theorem does not guarantee finding zeros of a polynomial function.
2. If synthetic division reveals a zero, why should we try that value again as a possible solution?

In Exercises 3 – 12, for the given polynomial, use the Rational Zeros Theorem to make a list of possible rational zeros.

3. $f(x)=x^3-2x^2-5x+6$
4. $f(x)=x^4+2x^3-12x^2-40x-32$
5. $f(x)=x^4-9x^2-4x+12$
6. $f(x)=x^3+4x^2-11x+6$
7. $f(x)=x^3-7x^2+x-7$
8. $f(x)=-2x^3+19x^2-49x+20$
9. $f(x)=-17x^3+5x^2+34x-10$
10. $f(x)=36x^4-12x^3-11x^2+2x+1$
11. $f(x)=3x^3+3x^2-11x-10$
12. $f(x)=2x^4+x^3-7x^2-3x+3$

In Exercises 13 – 32, find the real zeros of the polynomial. State the multiplicity of each real zero.

13. $f(x)=x^3-2x^2-5x+6$
14. $f(x)=x^4+2x^3-12x^2-40x-32$
15. $f(x)=x^4-9x^2-4x+12$
16. $f(x)=x^3+4x^2-11x+6$
17. $f(x)=x^3-7x^2+x-7$
18. $f(x)=-2x^3+19x^2-49x+20$
19. $f(x)=-17x^3+5x^2+34x-10$
20. $f(x)=36x^4-12x^3-11x^2+2x+1$
21. $f(x)=3x^3+3x^2-11x-10$
22. $f(x)=2x^4+x^3-7x^2-3x+3$
23. $f(x)=9x^3-5x^2-x$
24. $f(x)=6x^4-5x^3-9x^2$
25. $f(x)=x^4+2x^2-15$
26. $f(x)=x^4-9x^2+14$
27. $f(x)=3x^4-14x^2-5$
28. $f(x)=2x^4-7x^2+6$
29. $f(x)=x^6-3x^3-10$
30. $f(x)=2x^6-9x^3+10$
31. $f(x)=x^5-2x^4-4x+8$
32. $f(x)=2x^5+3x^4-18x-27$

In Exercises 33 – 42, find the real solutions of the polynomial equation.

33. $9x^3 = 5x^2 + x$

34. $9x^2 + 5x^3 = 6x^4$

35. $x^3 + 6 = 2x^2 + 5x$

36. $x^4 + 2x^3 = 12x^2 + 40x + 32$

37. $x^3 - 7x^2 = 7 - x$

38. $2x^3 = 19x^2 - 49x + 20$

39. $x^3 + x^2 = \dfrac{11x + 10}{3}$

40. $x^4 + 2x^2 = 15$

41. $14x^2 + 5 = 3x^4$

42. $2x^5 + 3x^4 = 18x + 27$

In Exercises 43 – 52, use Descartes' Rule of Signs (shown below) to list the possible number of positive and negative real zeros. Compare your results with solutions to Exercises 13 through 22.

Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicities) is one of the numbers $\{P, P-2, P-4, \ldots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicities) is one of the numbers $\{N, N-2, N-4, \ldots\}$.

Notes:

1. To determine variations in sign, consider $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. If we focus only on the **signs** of the coefficients, we start with a (+), followed by another (+), then switch to (–), and stay (–) for the remaining two coefficients. Since the signs of the coefficients switched **once** as we read from left to right, we say that $f(x)$ has **one** variation in sign.
2. The number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if $f(x)$ has 7 sign changes, then, counting multiplicities, f has 7, 5, 3, or 1 positive real zeros. If $f(-x)$ results in 4 sign changes, then, counting multiplicities, f has 4, 2, or 0 negative real zeros.

43. $f(x) = x^3 - 2x^2 - 5x + 6$

44. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$

45. $f(x) = x^4 - 9x^2 - 4x + 12$

46. $f(x) = x^3 + 4x^2 - 11x + 6$

47. $f(x) = x^3 - 7x^2 + x - 7$

48. $f(x) = -2x^3 + 19x^2 - 49x + 20$

49. $f(x) = -17x^3 + 5x^2 + 34x - 10$

50. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

51. $f(x) = 3x^3 + 3x^2 - 11x - 10$

52. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

53. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the *x*- and *y*-intercepts of the graph of f. Find the intervals on which the graph is above the *x*-axis and the intervals on which the graph is below the *x*-axis. Sketch a rough graph of f.

2.5 Complex Zeros of Polynomials

Learning Objectives

- Perform operations on complex numbers.
- Find all complex zeros of a polynomial.
- Factor a polynomial to linear and irreducible quadratic factors.
- Use the conjugate of a complex zero to identify an additional zero.
- Create a polynomial given information that includes complex zeros.

In **Section 2.4**, we focused on finding the real zeros of a polynomial function. In this section, we expand our horizons and look for the non-real zeros as well.

Complex Numbers

Consider the polynomial $p(x) = x^2 + 1$. The zeros of p are the solutions to $x^2 + 1 = 0$, or $x^2 = -1$. This equation has no real solutions, but x is a quantity whose square is -1. We write such a quantity as i, or $\sqrt{-1}$, and refer to it as the **imaginary unit**.

The imaginary unit i is a different kind of number with the property that $i^2 = -1$. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, and $(-2i)(3i) = -6i^2 = -6(-1) = 6$. The properties of i that distinguish it from the real numbers are listed below.

The **imaginary unit** i satisfies the following two properties.

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = \left(\sqrt{c}\right) \cdot i$

We have used the parentheses and the multiplication sign in part 2 for clarity. It is perfectly fine not to use either one; for example, $\sqrt{-c} = \sqrt{c}\, i$, as long as it does not cause any confusion. Of course, since the order of multiplication does not matter, you may also write $\sqrt{-c} = i\sqrt{c}$ to avoid the possible miswriting of i inside the square root.

Property 1, above, establishes that i does act as a square root[16] of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c, requiring that $c \geq 0$. For example, it is perfectly acceptable to say

$$\sqrt{-4} = \left(\sqrt{4}\right)i = 2i$$

However, $\sqrt{-(-4)} \neq \left(\sqrt{-4}\right)i$ or we would get

$$2 = \sqrt{4} = \sqrt{-(-4)} = \left(\sqrt{-4}\right)i = (2i)i = 2i^2 = 2(-1) = -2$$

which is false.[17] We are now in the position to define the **complex numbers**.

Definition 2.5. A **complex number** is defined as $a+bi$ where a and b are real numbers and i is the imaginary unit. The number a is the **real part** of the complex number and the number b is the **imaginary part** of the complex number.

Complex numbers include numbers like $3+2i$ and $\frac{2}{5}-\sqrt{3}i$. However, keep in mind that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, do not forget that the complex numbers include the real numbers; both 0 and $\pi-\sqrt{21}$ are considered complex numbers.

Example 2.5.1. Perform the indicated operations. Write your answer in the form $a+bi$.[18]

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $(3-4i)(3+4i)$
4. $\dfrac{1-2i}{3-4i}$
5. $\sqrt{-3}\sqrt{-12}$
6. $\sqrt{(-3)(-12)}$

Solution.

1. We combine like terms to get $(1-2i)-(3+4i) = 1-2i-3-4i = -2-6i$.
2. Using the distributive property, we get

$$\begin{aligned}(1-2i)(3+4i) &= (1)(3)+(1)(4i)-(2i)(3)-(2i)(4i)\\ &= 3+4i-6i-8i^2\\ &= 3-2i-8(-1)\\ &= 11-2i\end{aligned}$$

[16] Note the use of the indefinite article 'a' and that, while i is a square root of -1, $-i$ is the other square root of -1.

[17] We want to enlarge the number system so we can solve things like $x^2=-1$, but not at the cost of the established rules already set in place. For that reason, the general properties of radicals simply do not apply for even roots of negative quantities.

[18] Okay, we'll accept things like $3-2i$ even though it can be written as $3+(-2)i$.

3. Again using the distributive property, we have

$$\begin{aligned}(3-4i)(3+4i) &= (3)(3)+(3)(4i)-(4i)(3)-(4i)(4i)\\ &= 9-16i^2\\ &= 9-16(-1)\\ &= 25\end{aligned}$$

Notice that the product of these two complex numbers is a real number, and that $3+4i$ and $3-4i$ are of the forms $a+bi$ and $a-bi$, respectively, where $a=3$ and $b=4$. We define the **complex conjugate** of a number $a+bi$ to be $a-bi$, for $b\neq 0$. It can be shown that the product of any two complex conjugates, $a+bi$ and $a-bi$, is the real number a^2+b^2. (Try it!)

4. To simplify $\frac{1-2i}{3-4i}$, we deal with the denominator $3-4i$ by multiplying both the numerator and denominator by $3+4i$ (the conjugate of $3-4i$).

$$\begin{aligned}\frac{1-2i}{3-4i}\cdot\frac{3+4i}{3+4i} &= \frac{(1-2i)(3+4i)}{(3-4i)(3+4i)}\\ &= \frac{11-2i}{25} \qquad \text{See parts 2 and 3 of this example.}\\ &= \frac{11}{25}-\frac{2}{25}i\end{aligned}$$

5. We begin by using the second property of the imaginary unit to rewrite $\sqrt{-3}$ and $\sqrt{-12}$ in terms of i, and then apply the rules of radicals that are applicable to real radicals.

$$\begin{aligned}\sqrt{-3}\sqrt{-12} &= \left(\sqrt{3}\,i\right)\left(\sqrt{12}\,i\right)\\ &= \sqrt{3\cdot 12}\,i^2\\ &= \sqrt{36}(-1)\\ &= -6\end{aligned}$$

6. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)}=\sqrt{36}=6$.

□

Note that the conjugate of $2+7i$ is $2-7i$ while the conjugate of $2-7i$ is $2+7i$. This implies that $a+bi$ and $a-bi$ are conjugates of each other. A 'bar' notation can be used for the conjugate of $a+bi$ as follows: $\overline{a+bi}=a-bi$. Using bar notation, $\overline{4i}$ is $-4i$.

Example 2.5.2. Multiply $\left(x-(1+2i)\right)\left(x-(1-2i)\right)$ and simplify the result.

Solution. We use the distributive property to multiply, and then proceed by combining like terms.

$$\begin{aligned}\left(x-(1+2i)\right)\left(x-(1-2i)\right) &= x^2 - x(1-2i) - x(1+2i) + (1+2i)(1-2i) \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 \\ &= x^2 - 2x + 1 - 4(-1) \\ &= x^2 - 2x + 5\end{aligned}$$

□

Zeros of Polynomials

We now return to the business of zeros. Suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note that the quadratic does not factor nicely, so we resort to the Quadratic Formula.

$$\begin{aligned}x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i\end{aligned}$$

Two things are important to note. First, the zeros $1+2i$ and $1-2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with real coefficients, the zeros will be a complex conjugate pair. (Do you see why?) Next, recall that we found $\left(x-(1+2i)\right)\left(x-(1-2i)\right) = x^2 - 2x + 5$ in **Example 2.5.2**. This demonstrates that the Factor Theorem holds even for non-real zeros: $x = 1+2i$ is a zero of f and, sure enough, $\left(x-(1+2i)\right)$ is a factor of $f(x)$.

The Factor and Remainder Theorems hold for $(x-c)$, even if c is a complex number. But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is 'No', as long as the degree of the polynomial is at least one. The theorem that provides this answer is the Fundamental Theorem of Algebra. While the proof of the Fundamental Theorem of Algebra is saved for future mathematics classes, we use this important theorem, along with the Factor Theorem, to arrive at results involving complex factorization, included in the following theorem.

Theorem 2.6. The Fundamental Theorem of Algebra and Complex Factorization: Suppose f is a polynomial function with degree $n \geq 1$. Then f has at least one complex zero.

In actuality, f has exactly n zeros, counting multiplicities. If z_1, z_2, …, z_k are the distinct zeros of f, with multiplicities m_1, m_2, …, m_k, respectively, then $f(x) = a(x - z_1)^{m_1}(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$ and $m_1 + m_2 + \cdots + m_k = n$.

Note that the value a in **Theorem 2.6** is the leading coefficient of $f(x)$ (Can you see why?) and as such, we see that a polynomial is completely determined by its zeros, their multiplicities, and its leading coefficient. The following example demonstrates the results of **Theorem 2.6**.

Example 2.5.3. For the following polynomials, find the degree, the zeros, the multiplicity of each zero, and write the polynomial in factored form.

1. $f(x) = x - 2$
2. $g(x) = x^2 - 4x + 4$
3. $h(x) = 3x^3 + 12x$
4. $j(x) = x^4 - 16$

Solution.

1. We see that $f(x) = x - 2$ has degree 1, and cannot be further factored. By solving $x - 2 = 0$, we find that f has a zero, $x = 2$, of multiplicity 1.

2. The polynomial $g(x) = x^2 - 4x + 4$ has degree 2. We may rewrite g as $g(x) = (x - 2)^2$, from which we find one zero, $x = 2$, of multiplicity 2.

3. We note that $h(x) = 3x^3 + 12x$ has degree 3, and factoring results in $h(x) = 3x(x^2 + 4)$. We find that $x = 0$ is a zero, and then proceed to solve $x^2 + 4 = 0$.

$$\begin{aligned} x^2 + 4 &= 0 \\ x^2 &= -4 \\ x &= \pm\sqrt{-4} = \pm\sqrt{4}\,i = \pm 2i \end{aligned}$$

Thus, h has zeros $x = 0$, $x = 2i$, and $x = -2i$, each of multiplicity 1.

We say $h(x) = 3x(x^2 + 4)$ is factored **over the real numbers**. While not required to do so, we note that, by applying **Theorem 2.6**, we could further factor h as $h(x) = 3(x - 0)(x - (2i))(x - (-2i))$. In this case, we would say that h is factored **over the complex numbers**.

4. For $j(x) = x^4 - 16$, the degree is 4 and the polynomial factors as follows.

$$\begin{aligned} j(x) &= x^4 - 16 \\ &= (x^2 - 4)(x^2 + 4) \\ &= (x-2)(x+2)(x^2+4) \end{aligned}$$

The zeros are $x = 2$, $x = -2$, $x = 2i$, and $x = -2i$, each of multiplicity 1.

We note that $j(x) = (x-2)(x+2)(x^2+4)$ is factored over the real numbers. As in part 3, we could further factor j over the complex numbers as $j(x) = (x-2)(x-(-2))(x-(2i))(x-(-2i))$.

□

Complex Zeros

The last two results of this section will provide guidance in identifying zeros and factoring.

Theorem 2.7. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and $a+bi$ is a zero of f, then its complex conjugate, $a-bi$, is also a zero of f, or vice versa.

The proof of the Conjugate Pairs Theorem uses properties of conjugates, along with the assumption that $f(a+bi) = 0$, to show that $f(a-bi) = 0$. We leave this proof to the enthused reader for now.

Since nonreal zeros of a polynomial f come in conjugate pairs $a+bi$ and $a-bi$, the Factor Theorem kicks in to give us both $(x-(a+bi))$ and $(x-(a-bi))$ as factors of $f(x)$. This means that $(x-(a+bi))(x-(a-bi)) = x^2 - 2ax + (a^2+b^2)$ is an irreducible quadratic factor of f. As a result, we have our last theorem of this section.

Theorem 2.8. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored over the real numbers into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors corresponding to the nonreal zeros of f, with all factors having real number coefficients.

We now present an example that pulls together the major ideas of this section.

Example 2.5.4. Let $f(x) = x^5 - 3x^4 + 3x^3 - 5x^2 + 12$.

1. Find all of the zeros of f and state their multiplicities.
2. Factor f to linear and irreducible quadratic factors.

Solution.

1. We may start finding the zeros of f by looking for rational zeros. By the Rational Zeros Theorem, the possible rational zeros are ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12. Using synthetic division, we can check if any of these is a zero. Checking -1, we get

$$\begin{array}{r|rrrrrr} -1 & 1 & -3 & 3 & -5 & 0 & 12 \\ & \downarrow & -1 & 4 & -7 & 12 & -12 \\ \hline & 1 & -4 & 7 & -12 & 12 & 0 \end{array}$$

Thus, $x = -1$ is a zero and we can factor f as $f(x) = (x+1)(x^4 - 4x^3 + 7x^2 - 12x + 12)$. We continue the search for rational zeros of $x^4 - 4x^3 + 7x^2 - 12x + 12$.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 7 & -12 & 12 \\ & \downarrow & 2 & -4 & 6 & -12 \\ \hline 2 & 1 & -2 & 3 & -6 & 0 \\ & \downarrow & 2 & 0 & 6 & \\ \hline & 1 & 0 & 3 & 0 & \end{array}$$

Now we have $x^4 - 4x^3 + 7x^2 - 12x + 12 = (x-2)^2(x^2+3)$. The solutions to $x^2 + 3 = 0$ are $x = \pm\sqrt{3}i$. From **Theorem 2.6**, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $x = 2$ of multiplicity 2, and the zeros $x = -1$, $x = \sqrt{3}i$ and $x = -\sqrt{3}i$, each of multiplicity 1.

2. Following the synthetic division in part 1, we factor f as follows.

$$\begin{aligned} f(x) &= x^5 - 3x^4 + 3x^3 - 5x^2 + 12 \\ &= (x+1)(x^4 - 4x^3 + 7x^2 - 12x + 12) \\ &= (x+1)(x-2)^2(x^2+3) \end{aligned}$$

Since $x^2 + 3$ cannot be factored over the real numbers, it is an **irreducible quadratic** factor. Our final answer is $f(x) = (x+1)(x-2)^2(x^2+3)$.

□

Our next example uses **Theorem 2.7** to find missing zeros, and then uses these zeros to factor the polynomial in accordance with **Theorem 2.8**.

Example 2.5.5. Let $f(x)=x^4-x^3-2x^2-4x-24$.

1. Given that $x=2i$ is a zero of f, find the remaining zeros of f.
2. Factor f to linear and irreducible quadratic factors.

Solution.

1. Since f is a polynomial with real number coefficients and $x=2i$ is a zero of f, $x=-2i$ is also a zero by **Theorem 2.7**. Thus, the following is an irreducible quadratic factor of f:

$$\begin{aligned}(x-(2i))(x-(-2i)) &= (x-2i)(x+2i) \\ &= x^2-(2i)^2 \qquad (a-b)(a+b)=a^2-b^2 \\ &= x^2+4\end{aligned}$$

To find the remaining zeros, we make use of the factor x^2+4. Noting that $f(x)=(x^2+4)\cdot g(x)$ for some polynomial g, we have $g(x)=f(x)\div(x^2+4)$, or $(x^4-x^3-2x^2-4x-24)\div(x^2+4)$.

$$\begin{array}{r l}
 & x^2-\;x-\;6 \\
x^2+0x+4 \big) & \overline{x^4-\;x^3-2x^2-4x-24} \\
 & -\underline{(x^4+0x^3+4x^2)} \\
 & -x^3-6x^2-4x-24 \\
 & -\underline{(-x^3-0x^2-4x)} \\
 & -6x^2\qquad -24 \\
 & -\underline{(-6x^2\qquad -24)} \\
 & 0
\end{array}$$

That is, $f(x)\div(x^2+4)=x^2-x-6$, and so $f(x)=(x^2+4)(x^2-x-6)$. Solving $x^2-x-6=0$ gives us zeros of $x=-2$ and $x=3$, in addition to the zeros of $x=2i$ and $x=-2i$.

2. After part 1, there is not much factoring left to do. We already have $f(x)=(x^2+4)(x^2-x-6)$. Factoring x^2-x-6 gives us $f(x)=(x^2+4)(x-3)(x+2)$.

□

Our last example turns the tables and ask us to manufacture a polynomial with certain properties.

Example 2.5.6. Find a polynomial p of lowest degree that has integer coefficients and satisfies all of the following criteria.

- The graph of $y = p(x)$ touches, but does not cross, the *x*-axis at $(2,0)$.
- $x = 3i$ is a zero of p.
- As $x \to -\infty$, $p(x) \to -\infty$.
- As $x \to \infty$, $p(x) \to -\infty$.

Solution. Since the graph of p touches the *x*-axis at $(2,0)$, we know $x = 2$ is a zero of even multiplicity. We are looking for a polynomial of lowest degree, so we need $x = 2$ to have multiplicity of exactly 2. The Factor Theorem now tells us $(x-2)^2$ is a factor of $p(x)$.

Since $x = 3i$ is a zero and our final answer is to have integer (real) coefficients, $x = -3i$ is also a zero. The Factor Theorem kicks in again to give us $(x-3i)$ and $(x-(-3i)) = (x+3i)$ as factors of $p(x)$.

We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem, that $p(x) = a(x-2)^2(x-3i)(x+3i)$ for some real number a.

Our last concern is end behavior, for which we need the leading term. We expand p as follows.

$$\begin{aligned} p(x) &= a(x-2)^2(x-3i)(x+3i) \\ &= a\left(x^2-4x+4\right)\left(x^2+9\right) \\ &= a\left(x^2\right)\left(x^2\right)+\cdots \\ &= a\,x^4+\cdots \end{aligned}$$

The leading term of $p(x)$ is $a\,x^4$. For the end behavior $p(x) \to -\infty$ as $x \to \pm\infty$, we need $a < 0$. For simplicity, we choose $a = -1$. Then, $p(x) = (-1)(x-2)^2(x-3i)(x+3i)$. If we expand, we get $p(x) = -x^4 + 4x^3 - 13x^2 + 36x - 36$.

$\square$

The observant reader will note that we did not give any examples of applications that involve complex numbers. Students often wonder if complex numbers are used in 'real-world' applications. After all, we did call i the 'imaginary' unit. It turns out that complex numbers are useful in many applied fields such as fluid dynamics, electromagnetism and quantum mechanics, but most of the applications require mathematics well beyond College Algebra for a full understanding.

We invite you to find a few examples of complex number applications. A simple Internet search with the phrase ‘complex numbers in real life’ should get you started. Basic electronics classes are another place to look, but keep in mind that they might use the letter j where we have used i. For the remainder of the text, we will restrict our attention to real numbers. We do this primarily because the first Calculus course you will take, ostensibly the one that this text is preparing you for, studies only functions of real variables. We believe it makes more sense pedagogically to concentrate on concepts such as rational, exponential, and logarithmic functions.

2.5 Exercises

1. If $p(x)$ is a polynomial with real number coefficients and $-2i$ is a zero of $p(x)$, what do we know about the factors of $p(x)$?

2. If $p(x)$ is a polynomial with real number coefficients and no real zeros, can any conclusions be drawn about the end behavior of p?

In Exercises 3 – 12, use the given complex numbers z and w to find and simplify the following. Write your answers in the form $a+bi$.

(a) $z+w$ (b) $z \cdot w$ (c) z^2

(d) $\dfrac{z}{w}$ (e) the conjugate of z (f) $z \cdot (\text{the conjugate of } z)$

3. $z=2+3i$, $w=4i$
4. $z=1+i$, $w=-i$
5. $z=i$, $w=-1+2i$
6. $z=4i$, $w=2-2i$
7. $z=3-5i$, $w=2+7i$
8. $z=-5+i$, $w=4+2i$
9. $z=\sqrt{2}-\sqrt{2}i$, $w=\sqrt{2}+\sqrt{2}i$
10. $z=1-\sqrt{3}i$, $w=-1-\sqrt{3}i$
11. $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
12. $z=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $w=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

In Exercises 13 – 20, simplify the quantity.

13. $\sqrt{-49}$
14. $\sqrt{-9}$
15. $\sqrt{-25}\sqrt{-4}$
16. $\sqrt{(-25)(-4)}$
17. $\sqrt{-9}\sqrt{-16}$
18. $\sqrt{(-9)(-16)}$
19. $\sqrt{-(-9)}$
20. $-\sqrt{-9}$

We know that $i^2=-1$ which means $i^3=i^2 \cdot i=(-1)i=-i$ and $i^4=i^2 \cdot i^2=(-1)(-1)=1$. In Exercises 21 – 28, use this information to simplify the given power of i.

21. i^5
22. i^6
23. i^{-7}
24. i^8
25. i^{15}
26. i^{26}
27. i^{117}
28. i^{304}

In Exercises 29 – 34, find all zeros of the polynomial and then completely factor it to linear and irreducible quadratic factors.

29. $f(x) = x^3 + 3x^2 + 4x + 12$

30. $f(x) = x^3 - 2x^2 + 9x - 18$

31. $f(x) = 3x^3 - 13x^2 + 43x - 13$

32. $f(x) = x^3 + 6x^2 + 6x + 5$

33. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3$

34. $f(x) = 2x^4 - 7x^3 + 14x^2 - 15x + 6$

In Exercises 35 – 50, find all zeros of the polynomial.

35. $f(x) = x^2 - 4x + 13$

36. $f(x) = 3x^2 + 2x + 10$

37. $f(x) = x^2 - 2x + 5$

38. $f(x) = 9x^3 + 2x + 1$

39. $f(x) = 4x^3 - 6x^2 - 8x + 15$ (Hint: $x = -\frac{3}{2}$ is one of the zeros.)

40. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$ (Hint: $x = \frac{3}{2}$ is one of the zeros.)

41. $f(x) = 8x^4 + 50x^3 + 43x^2 + 2x - 4$ (Hint: $x = -\frac{1}{2}$ is one of the zeros.)

42. $f(x) = x^3 + 7x^2 + 9x - 2$

43. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18$

44. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5$

45. $f(x) = x^4 + 9x^2 + 20$

46. $f(x) = x^4 + 5x^2 - 24$

47. $f(x) = x^6 - 64$

48. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$

49. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)

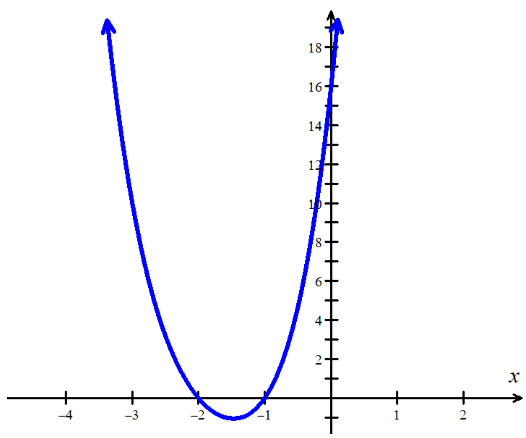
50. $f(x) = 2x^4 + 5x^3 + 13x^2 + 7x + 5$ (Hint: $x = -1 + 2i$ is a zero.)

In Exercises 51 – 55, create a polynomial f with real number coefficients that has all desired characteristics. You may leave the polynomial in factored form.

51. • The zeros of f are $x=\pm 1$ and $x=\pm i$.
    • The leading term of $f(x)$ is $42x^4$.
52. • $x=2i$ is a zero.
    • The point $(-1,0)$ is a local minimum on the graph of $y=f(x)$.
    • The leading term of $f(x)$ is $117x^4$.
53. • The solutions to $f(x)=0$ are $x=\pm 2$ and $x=\pm 7i$.
    • The leading term of $f(x)$ is $-3x^5$.
    • The point $(2,0)$ is a local maximum on the graph of $y=f(x)$.
54. • f is degree 5.
    • $x=6$, $x=i$ and $x=1-3i$ are zeros of f.
    • As $x\to-\infty$, $f(x)\to\infty$.
55. • The leading term of $f(x)$ is $-2x^3$.
    • $x=2i$ is a zero.
    • $f(0)=-16$.

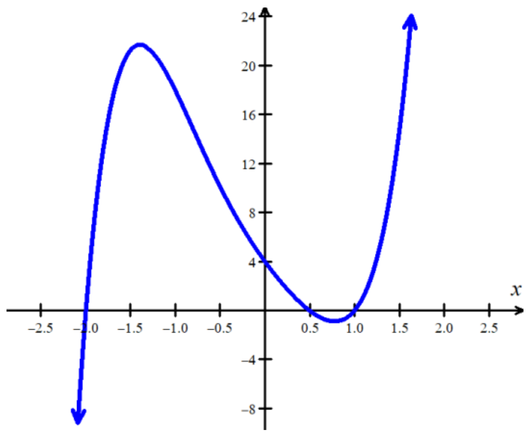
56. Find all zeros and completely factor $f(x)=x^4+7x^3+22x^2+32x+16$ to linear and irreducible quadratic factors. Give exact values, using radicals where necessary. The graph of $f(x)$ is below.

Figure Ex2.5. 1



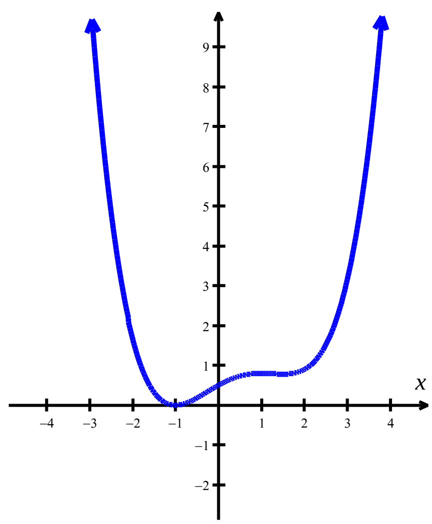
57. Find all zeros and completely factor $f(x) = 2x^5 + x^4 - x^3 + 4x^2 - 10x + 4$ to linear and irreducible quadratic factors. Give exact values, using radicals where necessary. The graph of $f(x)$ is shown below.

Figure Ex2.5. 2



58. Find a polynomial function $P(x)$ of degree 4 with real coefficients that has a zero of $x = 2 + i$ and a leading coefficient of $\frac{1}{10}$. A graph of $P(x)$ is shown below. Write the function $P(x)$ in factored form.

Figure Ex2.5. 3



2.6 Polynomial Inequalities

Learning Objectives

- Solve polynomial inequalities graphically.
- Solve polynomial inequalities analytically.

In this section, we develop techniques for solving polynomial inequalities. We determine solutions analytically, and look at them graphically. This first example motivates the core ideas.

Example 2.6.1. Let $f(x)=2x-1$ and $g(x)=5$.

1. Solve $f(x)=g(x)$.
2. Solve $f(x)<g(x)$.
3. Solve $f(x)>g(x)$.
4. Draw the graphs of $y=f(x)$ and $y=g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

Solution.

1. Replacing f and g by their definitions in the equation $f(x)=g(x)$, we have $2x-1=5$. Solving this equation, we get $x=3$.

2. The inequality $f(x)<g(x)$ is equivalent to $2x-1<5$. The solution of this inequality is $x<3$, or all numbers less than 3. In set-builder notation, the answer is $\{x \mid x<3\}$. The answer in interval notation is $(-\infty,3)$. The answer may be stated in either format, unless a specific format is required.

3. To find where $f(x)>g(x)$, we solve $2x-1>5$. We get $x>3$, or the interval $(3,\infty)$.

4. To draw the graph of $y=f(x)$, we plot $y=2x-1$, which is a line with a *y*-intercept of $(0,-1)$ and a slope of 2. The graph of $y=g(x)$ is the graph of $y=5$, which is a horizontal line through $(0,5)$.

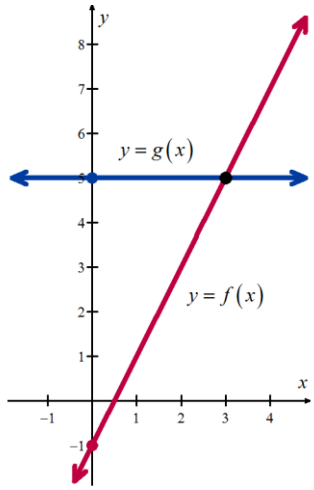
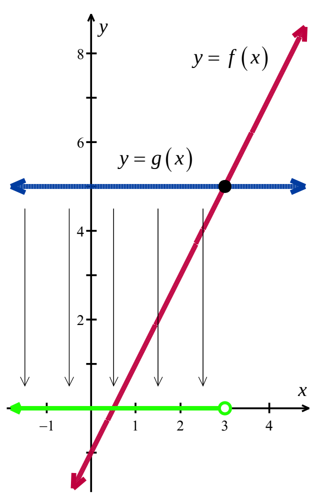


Figure 2.6. 1

In part 1, we found that $x=3$ is the solution to $f(x)=g(x)$. Noting that $f(3)=2(3)-1=5$ and $g(3)=5$, we confirm that the point $(3,5)$ is on both graphs. So, the solution to $f(x)=g(x)$ corresponds to the point of intersection of the graphs of $y=f(x)$ and $y=g(x)$.

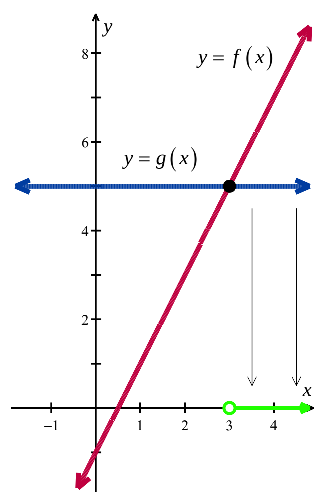
In part 2, the solution to $f(x)<g(x)$, which we found to be $x<3$, is all *x*-values for which $y=f(x)$ is less than $y=g(x)$. Graphically, the solution to $f(x)<g(x)$ corresponds to *x*-values for which the graph of f is below the graph of g. Similarly, the solution to $f(x)>g(x)$, in part 3, corresponds to *x*-values for which the graph of f is above the graph of g.

Figure 2.6. 2



$f(x)<g(x)$ on $(-\infty,3)$

Figure 2.6. 3



$f(x)>g(x)$ on $(3,\infty)$

□

Solving Polynomial Inequalities Graphically

The preceding example demonstrates the following.

Graphical Interpretation of Equations and Inequalities

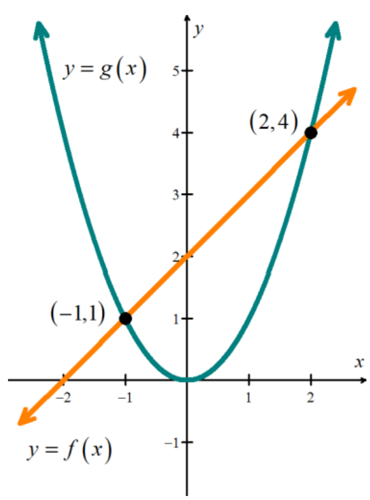
Suppose f and g are functions.

1. The solutions to $f(x)=g(x)$ are the x-values where the graphs of $y=f(x)$ and $y=g(x)$ intersect.
2. The solution to $f(x)<g(x)$ is the set of x-values where the graph of $y=f(x)$ is below the graph of $y=g(x)$.
3. The solution to $f(x)>g(x)$ is the set of x-values where the graph of $y=f(x)$ is above the graph of $y=g(x)$.

The next example turns the tables and furnishes the graphs of two functions from which we are asked to determine solutions to a corresponding equation and inequalities.

Example 2.6.2. The graphs of $f(x)=x+2$ and $g(x)=x^2$ are displayed below. Use these graphs to answer the following and explain what is being represented algebraically by the equation or inequality.

Figure 2.6. 4



1. Solve $f(x)=g(x)$. 2. Solve $f(x)>g(x)$. 3. Solve $f(x)\le g(x)$.

Solution.

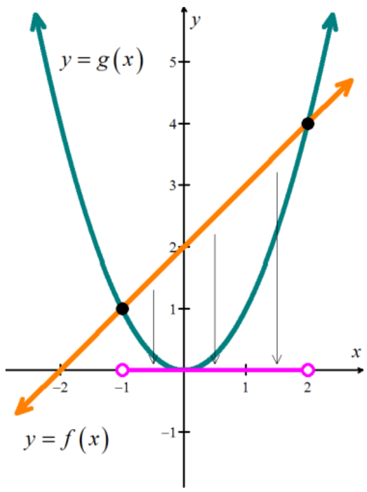
1. To solve $f(x)=g(x)$, we look for where the graphs of f and g intersect. This appears to be at the points $(-1,1)$ and $(2,4)$, in which case our solutions to $f(x)=g(x)$ are $x=-1$ and $x=2$.

The equation represented by $f(x)=g(x)$ is $x+2=x^2$ which, following this example, will be used to verify analytically that our graphical solution is correct.

2. To solve $f(x)>g(x)$, we look for where the graph of f is above the graph of g. This appears to happen for the *x*-values between -1 and 2. The solution is $\{x \mid -1<x<2\}$, or $(-1,2)$ in interval notation. (See **Figure 2.6.5** below.)

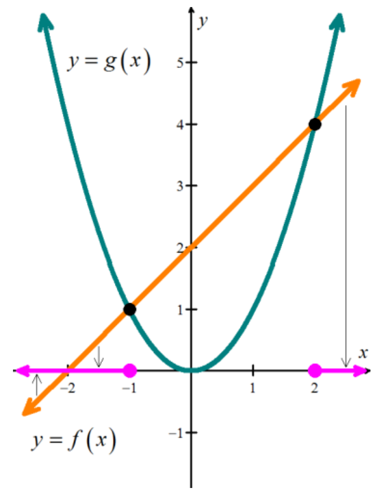
The inequality being represented by $f(x)>g(x)$ is $x+2>x^2$. We will verify shortly that the solution to this inequality is indeed $(-1,2)$.

Figure 2.6. 5



$f(x)>g(x)$

Figure 2.6. 6



$f(x)\le g(x)$

3. To solve $f(x)\le g(x)$, we look for solutions to $f(x)=g(x)$ as well as $f(x)<g(x)$. In part 1, we found the solution to the former equation to be $x=-1$ and $x=2$. To solve $f(x)<g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for *x*-values less than -1 and greater than 2. Hence, our solution to $f(x)\le g(x)$ is $(-\infty,-1]\cup[2,\infty)$. (See **Figure 2.6.6** above.)

This will be verified analytically, beginning with the observation that the inequality $f(x)\le g(x)$ represents $x+2\le x^2$.

□

Solving Polynomial Inequalities Analytically

We now look toward formulating a general analytic procedure for solving all polynomial inequalities.

In **Example 2.6.2**, for $f(x)=x+2$ and $g(x)=x^2$, we found a graphical solution to $f(x)=g(x)$. We now proceed to solve this equation algebraically.

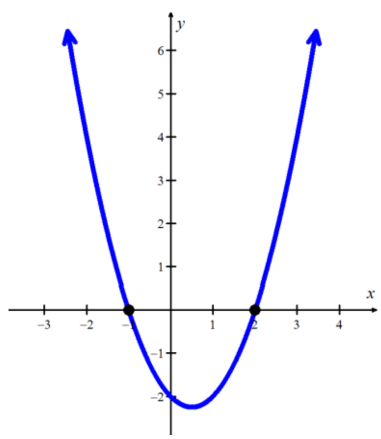
$$\begin{aligned} x+2&=x^2 \\ 0&=x^2-x-2 \\ 0&=(x-2)(x+1) \end{aligned}$$

The solutions are $x=-1$ and $x=2$, confirming our graphical solution.

To analytically determine the solution in **Example 2.6.2** to $f(x)>g(x)$, or $x+2>x^2$, we note that $x+2>x^2$ is equivalent to $x^2-x-2<0$. The solutions to the equation $x^2-x-2=0$ are $x=-1$ and $x=2$. These numbers divide the real numbers, the domain of the polynomial function $h(x)=x^2-x-2$, into three intervals: $(-\infty,-1)$, $(-1,2)$, and $(2,\infty)$.

Solving $x^2-x-2<0$ corresponds graphically to finding the values of x for which the graph of $h(x)=x^2-x-2$ (the parabola in the following illustration) is below the graph of $y=0$ (the *x*-axis).

Figure 2.6. 7

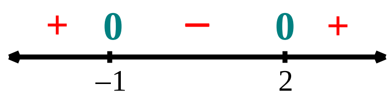


$y=x^2-x-2$

From this graph, we see immediately that the solution to $x^2-x-2<0$ is $-1<x<2$, or the interval $(-1,2)$.

Analytically, in each of the intervals $(-\infty,-1)$, $(-1,2)$, and $(2,\infty)$, the polynomial $h(x)=x^2-x-2$, being a continuous function, must have the same sign since, by the Intermediate Value Theorem, if the sign changes there must be an additional zero in that interval. (The only zeros are $x=-1$ and $x=2$.) To determine the sign of h in each interval, we just need to check its value at a number in that interval. We choose one number in each of the three intervals: $x=-2$ in $(-\infty,-1)$, $x=0$ in $(-1,2)$, and $x=3$ in $(2,\infty)$. We find $h(-2)=(-2)^2-(-2)-2=4>0$, $h(0)=(0)^2-(0)-2=-2<0$, and $h(3)=(3)^2-(3)-2=4>0$. Therefore, $x^2-x-2>0$ for all x in $(-\infty,-1)$ and $(2,\infty)$; $x^2-x-2<0$ for all x in $(-1,2)$. We can schematically represent this with the following **sign diagram**.

Figure 2.6. 8



Here, the '+' above a portion of the number line indicates $x^2-x-2>0$ for values of x in that interval; the '−' indicates $x^2-x-2<0$ for the corresponding values of x. The numbers labeled on the number line are the zeros of $y=x^2-x-2$, so we place '**0**' above them. We see that the solution to $x^2-x-2<0$ is $(-1,2)$.

Similarly, in solving $f(x)\le g(x)$, which represents $x+2\le x^2$, and is equivalent to $x^2-x-2\ge 0$, we see from the number line that the solution, which includes the zeros, is $(-\infty,-1]\cup[2,\infty)$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function.

Steps for Solving a Polynomial Inequality

1. Rewrite the inequality, if necessary, as a polynomial function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the real zeros of f and place them on the number line with the number 0 above them.
3. Choose a number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the interval(s) that correspond to the correct sign to solve the inequality.

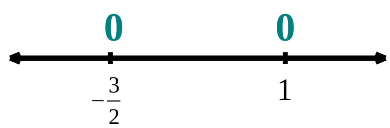
Example 2.6.3. Solve the following inequalities. Confirm your answer graphically.

1. $2x^2 \le 3-x$
2. $x^2-2x>1$
3. $x^2+1\le 2x$
4. $2x^3-19x^2+49x-20<0$

Solution.

1. To solve $2x^2 \le 3-x$, we rewrite it as $2x^2+x-3\le 0$, in order to have a 0 on the right side. We find the zeros of $f(x)=2x^2+x-3$ by solving $2x^2+x-3=0$ for x. Factoring gives $(2x+3)(x-1)=0$, for zeros of $x=-\dfrac{3}{2}$ and $x=1$.

Figure 2.6. 9

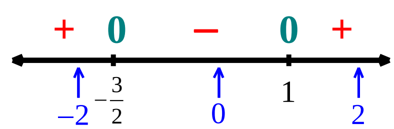


Next, we choose a test value in each of the resulting intervals: $\left(-\infty,-\dfrac{3}{2}\right)$, $\left(-\dfrac{3}{2},1\right)$, and $(1,\infty)$.

For the interval $\left(-\infty,-\dfrac{3}{2}\right)$, we choose[19] $x=-2$; for $\left(-\dfrac{3}{2},1\right)$ we pick $x=0$; for $(1,\infty)$, $x=2$ is our test value. Evaluating the function at the three test values gives the following.

$$f(-2)=3 \quad >0$$
$$f(0)=-3 \quad <0$$
$$f(2)=7 \quad >0$$

Figure 2.6. 10

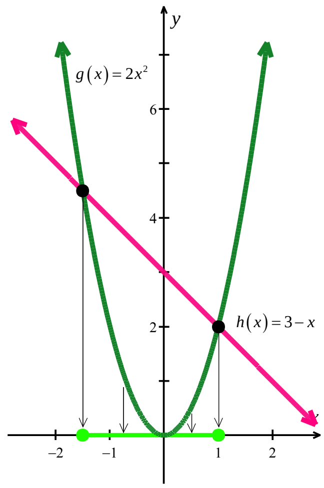


Since we are solving $2x^2+x-3\le 0$, we look for solutions to $2x^2+x-3<0$ as well as $2x^2+x-3=0$. Therefore, the solution is $-\dfrac{3}{2}\le x\le 1$, or the interval $\left[-\dfrac{3}{2},1\right]$.

We next view our solution graphically, while noting that a graphical solution is only an estimate. The solution of $2x^2\le 3-x$ is all x-values where the graph of $g(x)=2x^2$ is either below the graph of $h(x)=3-x$ or intersects it. Alternatively, we can look for the x-values where the graph of $y=2x^2+x-3$ lies below the x-axis, $y=0$, or intersects it.

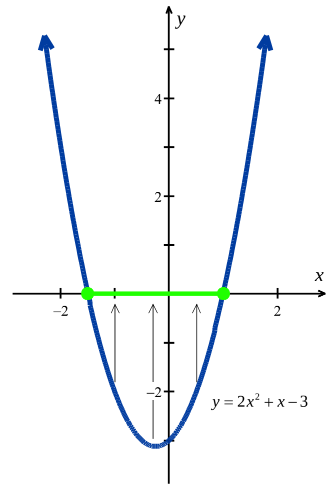
[19] We have to choose something in each interval, but any number in the interval will result in the same sign chart.

Figure 2.6. 11



$g(x) \le h(x)$ for x values in $\left[-\frac{3}{2}, 1\right]$

Figure 2.6. 12



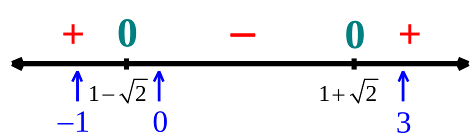
$2x^2 + x - 3 \le 0$ for x values in $\left[-\frac{3}{2}, 1\right]$

In either case, we verify that the answer is the interval $\left[-\frac{3}{2}, 1\right]$.

2. We re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and let $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find that the quadratic expression $x^2 - 2x - 1$ does not factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$ and arrive at $x = 1 \pm \sqrt{2}$. These zeros divide the number line into three pieces. To help us decide on test values, we approximate the values of zeros: $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$, and $x = 3$ as our test values.

$$\begin{aligned} f(-1) &= 2 \quad > 0 \\ f(0) &= -1 \quad < 0 \\ f(3) &= 2 \quad > 0 \end{aligned}$$

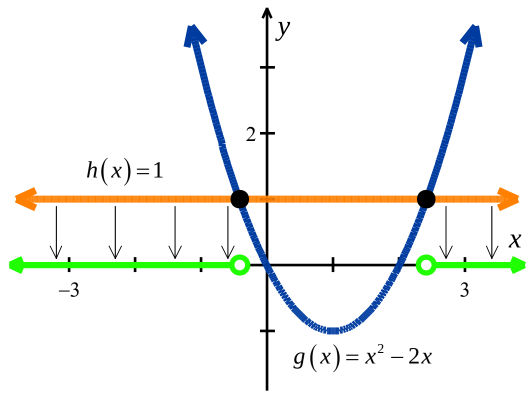
Figure 2.6. 13



Our solution to $x^2 - 2x - 1 > 0$ is where we have '+', which is $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x-values where the graph of g is above the graph of h.

Figure 2.6. 14

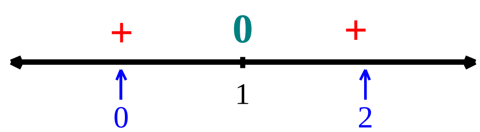


$g(x)>h(x)$ for x values in $\left(-\infty,1-\sqrt{2}\right)\cup\left(1+\sqrt{2},\infty\right)$

3. We rewrite $x^2+1\le 2x$ as $x^2-2x+1\le 0$ and let $f(x)=x^2-2x+1$. Solving $x^2-2x+1=0$, we find the only zero of f to be $x=1$. This zero divides the number line into two intervals, from which we choose $x=0$ and $x=2$ as test values.

$$f(0)=1 \quad >0$$
$$f(2)=1 \quad >0$$

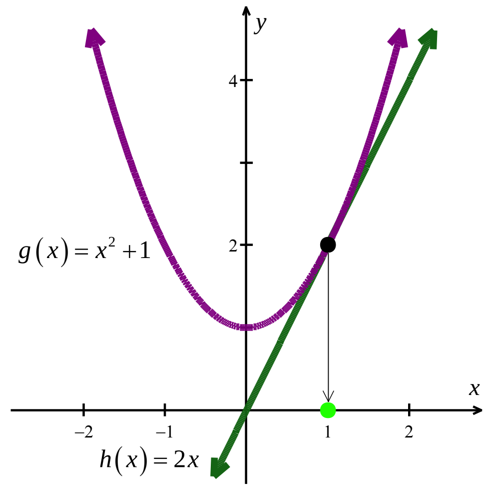
Figure 2.6. 15



Since we are looking for solutions to $x^2-2x+1\le 0$, we are looking for *x*-values where $x^2-2x+1<0$ as well as where $x^2-2x+1=0$. Looking at our sign diagram, there are no places where $x^2-2x+1<0$ since there is no '–'. Our only solution is $x=1$, where $x^2-2x+1=0$. We can also write this solution as $\{1\}$.

Graphically, we solve $x^2+1\le 2x$ by graphing $g(x)=x^2+1$ and $h(x)=2x$. We are looking for the *x*-values where the graph of g is below the graph of h for solutions to $x^2+1<2x$ and points where the two graphs intersect for solutions to $x^2+1=2x$.

Figure 2.6. 16



$g(x) \le h(x)$ when $x = 1$

Notice that the line and the parabola touch at $(1,2)$, but the parabola is always above the line otherwise.[20]

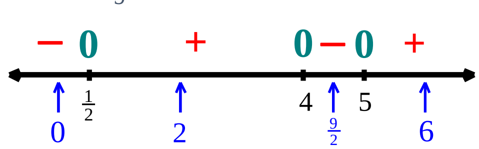
4. To solve our last inequality, $2x^3 - 19x^2 + 49x - 20 < 0$, we set $f(x) = 2x^3 - 19x^2 + 49x - 20 = 0$ and use methods from **Section 2.4** to find the zeros. We find that $x = 4$ is a zero through synthetic division, after which we factor the function.

$$\begin{array}{r|rrrr} 4 & 2 & -19 & 49 & -20 \\ & \downarrow & 8 & -44 & 20 \\ \hline & 2 & -11 & 5 & 0 \end{array}$$

We find $f(x) = (x-4)(2x^2 - 11x + 5) = (x-4)(2x-1)(x-5)$. Thus, $x = \frac{1}{2}$ and $x = 5$ are the remaining zeros. In the resulting intervals of $\left(-\infty, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 4\right)$, $(4,5)$, and $(5,\infty)$, we choose test values of $x = 0$, $x = 2$, $x = \frac{9}{2}$ and $x = 6$, respectively.

$$\begin{aligned} f(0) &= -20 && < 0 \\ f(2) &= 18 && > 0 \\ f\left(\frac{9}{2}\right) &= -2 && < 0 \\ f(6) &= 22 && > 0 \end{aligned}$$

Figure 2.6. 17

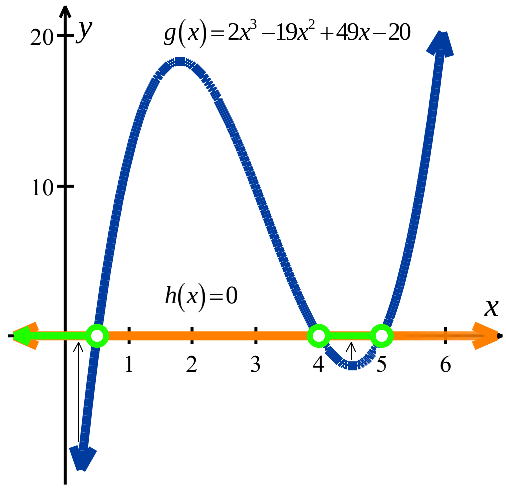


[20] In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1,2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

Our solution to $2x^3-19x^2+49x-20<0$ is where we have '$-$', which is $\left(-\infty,\frac{1}{2}\right)\cup(4,5)$.

Graphically, we solve $2x^3-19x^2+49x-20<0$ by graphing $g(x)=2x^3-19x^2+49x-20$ and $h(x)=0$. We are looking for *x*-values where the graph of g is below the graph of h.

Figure 2.6. 18



$g(x)<h(x)$ for x values in $\left(-\infty,\frac{1}{2}\right)\cup(4,5)$

□

2.6 Exercises

1. In creating a sign chart, how many test values must be used to determine the solution of an inequality whose highest power term is of degree 3?

2. If a polynomial inequality has no solutions, what can be said about the graphs of the two sides of the inequality?

In Exercises 3 – 26, solve the inequality. Write your answer using interval notation.

3. $(x-7)(x+2)<0$

4. $-3(x+4)(x+5)\geq 0$

5. $x^2+2x-3\geq 0$

6. $16x^2+8x+1>0$

7. $x^2+9<6x$

8. $9x^2+16\geq 24x$

9. $x^2+4\leq 4x$

10. $x^2+1<0$

11. $3x^2\leq 11x+4$

12. $x>x^2$

13. $2x^2-4x-1>0$

14. $5x+4\leq 3x^2$

15. $x^2+x+1\geq 0$

16. $(x-9)(x+11)(x-7)^2<0$

17. $-2(x-3)(x-1)(x+7)\leq 0$

18. $x^4-9x^2\leq 4x-12$

19. $(x-1)^2\geq 4$

20. $4x^3\geq 3x+1$

21. $x^4\leq 16+4x-x^3$

22. $3x^2+2x<x^4$

23. $\dfrac{x^3+2x^2}{2}<x+2$

24. $\dfrac{x^3+20x}{8}\geq x^2+2$

25. $2x^4>5x^2+3$

26. $x^6+x^3\geq 6$

27. The profit, in dollars, made by selling x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is given by $P(x)=-x^2+25x-100$, for $0\leq x\leq 35$. How many bottles of tonic must be sold to make at least \$50 in profit?

28. Suppose $C(x)=x^2-10x+27$, $3\leq x\leq 15$, represents the marginal cost in hundreds of dollars to produce an additional x thousand pens. Find how many additional pens can be produced with marginal cost of no more than \$1100. Give your answer as an interval of values.

29. Suppose $d(x)=0.04x^2+0.6x$ represents the stopping distance in feet of a car from the speed of x miles per hour. Find the maximum speed from which the car stops in no more than 130 feet.

30. The temperature T, in degrees Fahrenheit, t hours after 6 AM, is given by $T(t)=-\frac{1}{2}t^2+8t+32$ for $0\le t\le 12$. When is it warmer than 42° Fahrenheit?

31. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t)=-5t^2+100t$ for $0\le t\le 20$. When is the rocket at least 250 feet off the ground? Round your answer to two decimal places.

32. Let $f(x)=3x^{10}-7x^9-6x^8+30x^7-19x^6-29x^5+34x^4+6x^3-12x^2$. Use the fact that $f(1-i)=0$ to solve the inequality $f(x)\ge 0$. (Hint: Start with finding rational zeros.)

CHAPTER 3
RATIONAL FUNCTIONS

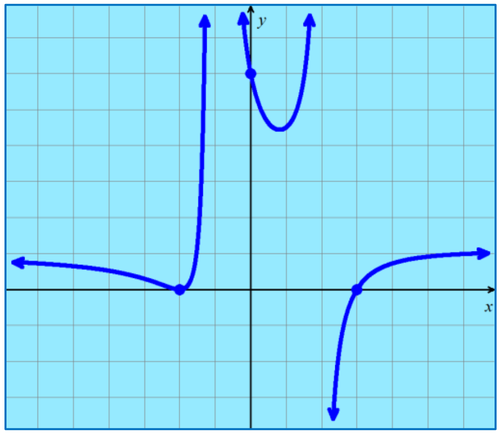


Figure 3.0. 1

Chapter Outline

Introduction

In this chapter, you will explore rational functions numerically, analytically, and graphically. A primary focus throughout the chapter will be the domain of a rational function; what is happening near values excluded from the domain, if any, and what is happening for ever larger positive or negative input values. The goal is for you to make sense of rational functions numerically, analytically, and graphically. You will begin by evaluating the function near values excluded from the domain, if any, and for very large positive or negative input values to get a sense of what the function is doing. On this foundation, you will develop an understanding of vertical asymptotes, holes, and horizontal (end behavior) asymptotes. Using domain, *x*- and *y*-intercepts, vertical asymptotes, holes, end behavior (horizontal or slant asymptotes), you will sketch graphs of rational functions. In the last section, you will use both graphical and analytical methods to solve rational equations and inequalities.

Section 3.1 is devoted to introducing you to rational functions and helping you understand their behavior near values excluded from the domain and for ever larger positive or negative input values. The section starts with a definition of a rational function. Next there is a review of how to determine the domain of a rational function. Using numeric strategies, we then explore how some rational functions behave near values excluded from the domain; this leads to the idea of a vertical asymptote. A similar numeric process for large positive and negative input values is employed to develop the idea of end behavior/horizontal asymptotes. You also review how to find *x*- and *y*- intercepts. You will continue to develop skills throughout the section with progressively more ‘complicated’ functions. The end goal is for you to be able to describe behavior of rational functions as a step toward sketching them in the next couple of sections. You will also rely on these understandings for analytic purposes such as solving equations or inequalities later on.

Section 3.2 builds from Section 3.1 by asking you to use your understanding of domain, vertical and horizontal asymptotes, and *x*- and *y*-intercepts to graph rational functions. The section provides several examples of functions with more than one vertical asymptote. Throughout the section, you are required to analyze and synthesize your understanding in order to create accurate graphs of irreducible rational functions with vertical and horizontal asymptotes.

Section 3.3 introduces both slant asymptotes and holes to further your understanding of behavior and graphing of rational functions. You will see, for the first time, rational functions where a factor in the denominator reduces with one in the numerator (commonly referred to as ‘canceling’). You will learn that while you must determine the domain of a rational function before you reduce, you find intercepts, asymptotes, and proceed with graphing the function after reducing. You will find that exclusions in the domain that ‘cancel out’ create ‘holes’ in the graph of the function (not asymptotes). This section also has rational functions where the degree of the numerator is larger than that of the denominator. Again, you will use the strategy of evaluating functions for very large positive and negative values of x to determine end behavior (slant asymptotes).

Section 3.4 is devoted to solving rational equations and inequalities. The section starts by making visual sense of a rational equation or inequality by graphing the left and right side of the equation or inequality, to help you ‘see’ the solution or solution region. You then build on that understanding to learn algebraic methods for solving.

3.1 Introduction to Rational Functions

Learning Objectives

- Identify a rational function.
- Determine the domain of a rational function.
- Find the *x*- and *y*-intercepts for a rational function.
- Identify vertical and horizontal asymptotes.
- Graph irreducible rational functions with constant or first degree numerators and denominators of degree one.

In this chapter, we study **rational functions** – functions that are ratios of polynomial functions.

Functions that are Rational

Definition 3.1. A **rational function** is a function that is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form $r(x)=\dfrac{p(x)}{q(x)}$, where p and q are polynomial functions[1], and q is not the zero polynomial.

Example 3.1.1. Determine if the following functions are rational functions. Explain your reasoning.

1. $f(x)=\dfrac{2x-1}{x+1}$
2. $g(x)=\dfrac{2\sqrt{x}-4x+3}{3x^2-5x+2}$

Solution.

1. $f(x)=\dfrac{2x-1}{x+1}$ is of the form $f(x)=\dfrac{p(x)}{q(x)}$ where $p(x)=2x-1$ and $q(x)=x+1$ are polynomial functions. So, f is a ratio of two polynomial functions and hence a rational function.
2. $g(x)=\dfrac{2\sqrt{x}-4x+3}{3x^2-5x+2}$ is of the form $g(x)=\dfrac{p(x)}{q(x)}$ with $p(x)=2\sqrt{x}-4x+3$. However, $p(x)=2\sqrt{x}-4x+3=2x^{\frac{1}{2}}-4x+3$ is not a polynomial function since x is raised to a non-integer power in the first term. Thus, g is not a rational function.

□

[1] According to this definition, all polynomial functions are also rational functions. (Take $q(x)=1$.)

Domains of Rational Functions

Recall that a fraction is defined only if its denominator is not zero. To find the domain of the function $f(x)=\dfrac{2x-1}{x+1}$, from the previous example, we start with the set of real numbers, then find the real zeros of the denominator and exclude them from the set of real numbers. Solving $x+1=0$ results in $x=-1$. Hence, the domain is all real numbers except -1, or $\{x \mid x \neq -1\}$. In interval notation, this is $(-\infty,-1)\cup(-1,\infty)$. In the following example, we determine the domain of three rational functions that appear similar on first glance, but that each have unique characteristics.

Example 3.1.2. Find the domain of the following rational functions.

1. $F(x)=\dfrac{x+1}{x^2-9}$
2. $G(x)=\dfrac{x+1}{x^2+9}$
3. $H(x)=\dfrac{x+3}{x^2-9}$

Solution.

1. To find the domain of $F(x)=\dfrac{x+1}{x^2-9}$, we begin by setting the denominator equal to zero.

$$\begin{aligned} x^2-9&=0 \\ (x-3)(x+3)&=0 \end{aligned}$$

   We find that $x=\pm 3$ results in a denominator of zero, so our domain is $\{x \mid x \neq \pm 3\}$, or $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

2. Proceeding as before, we determine the domain of $G(x)=\dfrac{x+1}{x^2+9}$ by first setting the denominator equal to zero. Since there are no real solutions to $x^2+9=0$, there are no values of x that must be excluded from the set of real numbers. Thus, the domain is all real numbers, $(-\infty,\infty)$.

3. The domain of $H(x)=\dfrac{x+3}{x^2-9}$ is identical to the domain of $F(x)=\dfrac{x+1}{x^2-9}$, since their denominators are the same. From part 1, the domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

□

In part 3 of the preceeding example, we must avoid the temptation to simplify $H(x)=\dfrac{x+3}{x^2-9}$ as $H(x)=\dfrac{x+3}{(x+3)(x-3)}=\dfrac{1}{x-3}$. Without noting that cancellation can only be done if $x+3\neq 0$, such a simplification will result in an incorrect domain. We will look more closely at rational functions that contain common factors in the numerator and denominator in a later section.

Finding Intercepts

An *x*-intercept is a point on the graph of a function where the input value of x gives an output function value of zero. For rational functions, this occurs when the input value of x results in a numerator of zero, as long as x is in the domain of the function (the denominator is not zero). The *y*-intercept for a rational function occurs where the input value is $x=0$, if the function is defined at $x=0$. If the function is not defined at $x=0$, there is no *y*-intercept.

Example 3.1.3. Find the *x*- and *y*-intercepts of the graphs of the following three functions.

1. $F(x)=\dfrac{x+1}{x^2-9}$ 2. $K(x)=\dfrac{x+3}{x^2+9}$ 3. $y=\dfrac{-2}{x^2-9}$

Solution.

1. To find the *x*-intercepts for $y=F(x)=\dfrac{x+1}{x^2-9}$, we solve $y=0$ and recall that a rational function equals zero when the numerator is zero.

$$\frac{x+1}{x^2-9}=0$$
$$x+1=0 \quad \text{must have } x^2-9\neq 0.$$
$$x=-1$$

In **Example 3.1.2**, we found that the domain of F is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$, which includes $x=-1$. Thus, the *x*-intercept is the point $(-1,0)$. To find the *y*-intercept, we set $x=0$ and solve for y:

$$y=F(0)=\frac{0+1}{0^2-9}=-\frac{1}{9}$$

The *y*-intercept is the point $\left(0,-\dfrac{1}{9}\right)$.

2. For $y=K(x)=\dfrac{x+3}{x^2+9}$, *x*-intercepts are found by setting $y=0$:

$$0=\frac{x+3}{x^2+9}$$
$$x+3=0 \quad \text{must have } x^2+9\neq 0$$
$$x=-3$$

The domain of K, which is all real numbers, includes $x=-3$. Thus, we have an *x*-intercept at the point $(-3,0)$. We find the *y*-intercept by setting $x=0$:

$$y=K(0)=\frac{0+3}{0^2+9}=\frac{3}{9}=\frac{1}{3}$$

The y-intercept is the point $\left(0,\ \frac{1}{3}\right)$.

3. Since the numerator of $y=\frac{-2}{x^2-9}$ is −2, which is never equal to zero, the graph of y does not have any x-intercepts. Setting $x=0$ results in $y=\frac{-2}{0^2-9}=\frac{2}{9}$, so the y-intercept is the point $\left(0,\ \frac{2}{9}\right)$.

□

Identifying Vertical Asymptotes

Consider the function $f(x)=\frac{2x-1}{x+1}$ from **Example 3.1.1**. We found that $x=-1$ is not in the domain of f, which means $f(-1)$ is undefined. To find out more about the **local behavior** of f near $x=-1$, we can make two tables that show corresponding function values when x is close to -1.

We first choose values a little less than -1; for example, $x=-1.1$, $x=-1.01$, $x=-1.001$, and so on. These values are to the left of -1 on the real number line, so we say they 'approach -1 from the left.'

x	-1.1	-1.01	-1.001	-1.0001
$f(x)=\frac{2x-1}{x+1}$	32	302	3002	30002

As the x-values approach -1 from the left, the function values become larger and larger positive numbers. We express this symbolically by stating

$$\text{as } x\to-1^-,\ f(x)\to\infty$$

The superscript '−' indicates that the x-values approach -1 from the negative side of the real number line, or the left side of -1. Likewise, the values $x=-0.9$, $x=-0.99$, $x=-0.999$, etc., are to the right of -1 on the real number line, so we say these values 'approach -1 from the right.'

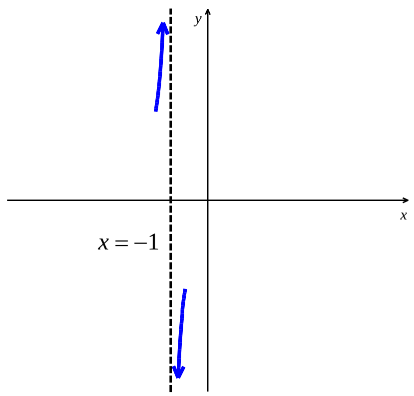
x	-0.9	-0.99	-0.999	-0.9999
$f(x)=\frac{2x-1}{x+1}$	-28	-298	-2998	-29998

As the x values approach -1 from the right, the function values become very large negative numbers, expressed symbolically by

$$\text{as } x\to-1^+,\ f(x)\to-\infty$$

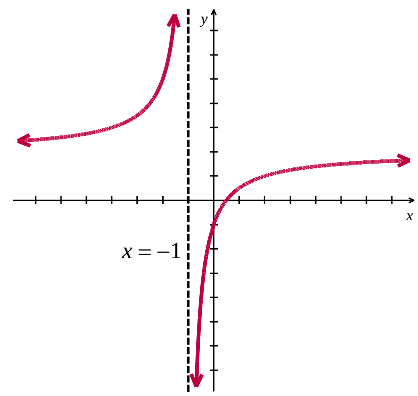
The superscript '+' indicates that the x-values approach -1 from the positive side of the real number line, or the right side of -1. These numerical results are confirmed by the following graph of $y=f(x)$. For this type of unbounded behavior, we say the graph has a **vertical asymptote** of $x=-1$, which is drawn as a dashed line.

Figure 3.1. 1



Behavior near $x=-1$:
as $x \to -1^-$, $f(x) \to \infty$; as $x \to -1^+$, $f(x) \to -\infty$

Figure 3.1. 2



$$y=f(x)=\frac{2x-1}{x+1}$$

Definition 3.2. The line $x=c$ is called a **vertical asymptote** of the graph of a function $y=f(x)$ if, as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

The following steps can be used to identify vertical asymptotes.

Identifying Vertical Asymptotes

1. Reduce the fraction, if possible. To reduce the fraction, factor the numerator and denominator and cancel factors appearing in both the numerator and denominator.[2]
2. Set the denominator equal to zero and solve the resulting equation.
3. If c is a real zero of the denominator, then $x=c$ is a vertical asymptote.[3]

For practice in finding vertical asymptotes, we return to the functions from **Example 3.1.2**.

Example 3.1.4. Identify the vertical asymptotes of the graphs of the following rational functions.

1. $F(x)=\dfrac{x+1}{x^2-9}$ 2. $G(x)=\dfrac{x+1}{x^2+9}$ 3. $H(x)=\dfrac{x+3}{x^2-9}$

Solution.

1. We first factor the numerator and denominator of $F(x)=\dfrac{x+1}{x^2-9}$ to get $F(x)=\dfrac{x+1}{(x+3)(x-3)}$.

Since there are no common factors, this fraction cannot be reduced. We next set the denominator

[2] Keep in mind that the domain should be carefully determined before canceling any common factors.
[3] There may be one, more than one, or no vertical asymptotes.

equal to zero to arrive at $x+3=0$ and $x-3=0$, from which $x=\pm 3$. Hence, the lines $x=-3$ and $x=3$ are vertical asymptotes.

2. Since the fraction $G(x)=\dfrac{x+1}{x^2+9}$ is irreducible, we proceed to set the denominator equal to zero. As noted in **Example 3.1.2**, there are no real solutions to $x^2+9=0$, so the graph of $y=G(x)$ does not have any vertical asymptotes.

3. To reduce the fraction $H(x)=\dfrac{x+3}{x^2-9}$, we factor the numerator and the denominator, and then cancel the common factor:[4]

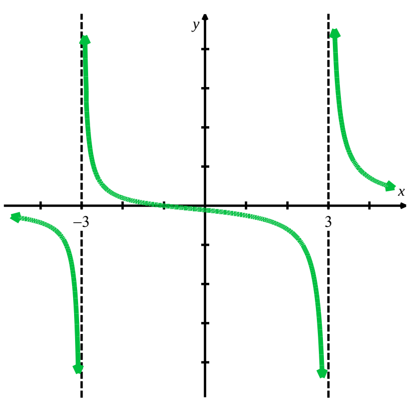
$$\frac{x+3}{x^2-9}=\frac{x+3}{(x-3)(x+3)}=\frac{1}{x-3},\quad x\neq -3$$

Setting the denominator equal to zero, we have $x-3=0$ and get $x=3$. The vertical asymptote is the line $x=3$.

□

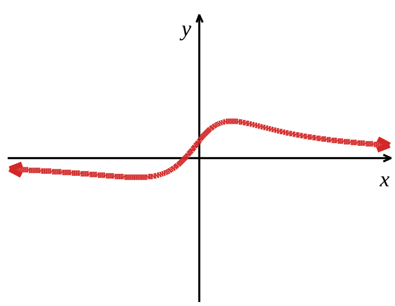
Using graphing technology[5] enables us to visualize the results from **Example 3.1.4**.

Figure 3.1. 3



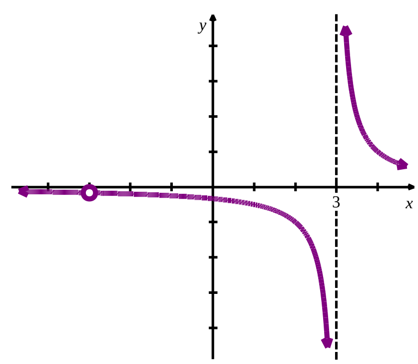
$y=F(x)=\dfrac{x+1}{x^2-9}$

Figure 3.1. 4



$y=G(x)=\dfrac{x+1}{x^2+9}$

Figure 3.1. 5



$y=H(x)=\dfrac{x+3}{x^2-9}$

Identifying Horizontal Asymptotes

Now, let us consider the behavior of the graph of the function $f(x)=\dfrac{2x-1}{x+1}$ for large positive and negative x-values, referred to as the **end behavior** of the graph. As we discussed in **Section 2.2**, the end

[4] We will look at the effect that this factor, $x+3$, has on the graph when we get to **Section 3.3**.
[5] Try Wolfram Alpha, Desmos, GeoGebra, or a free online graphing calculator.

behavior of a function is its behavior as x attains larger[6] and larger negative values without bound, denoted $x \to -\infty$, and as x becomes large without bound, written as $x \to \infty$. We again refer to tables.

x	-10	-100	-1000	-10000
$f(x)=\dfrac{2x-1}{x+1}$	≈ 2.333	≈ 2.0303	≈ 2.0030	≈ 2.0003

Values of $f(x)$ as x approaches $-\infty$

x	10	100	1000	10000
$f(x)=\dfrac{2x-1}{x+1}$	≈ 1.7273	≈ 1.9703	≈ 1.9970	≈ 1.9997

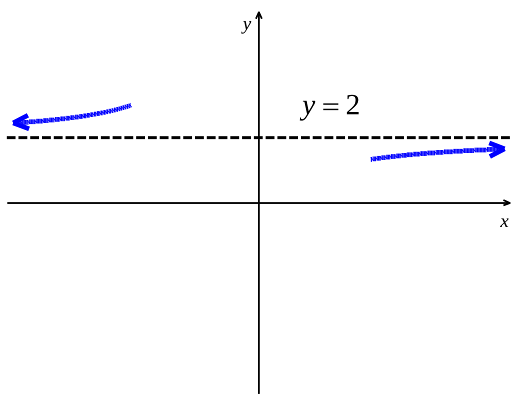
Values of $f(x)$ as x approaches ∞

From these tables, it appears that the value of $f(x)$ approaches 2 for both large positive and negative x-values. We express this symbolically as follows:

$$\text{as } x \to -\infty,\ f(x) \to 2, \text{ and as } x \to \infty,\ f(x) \to 2.$$

This means that, as shown below, the graph of $f(x)=\dfrac{2x-1}{x+1}$ 'levels off' and approaches the horizontal line $y=2$ on both the left and right sides of the graph. In this case, we say the graph of $y=f(x)$ has a **horizontal asymptote** of $y=2$, drawn as follows by a dashed line.

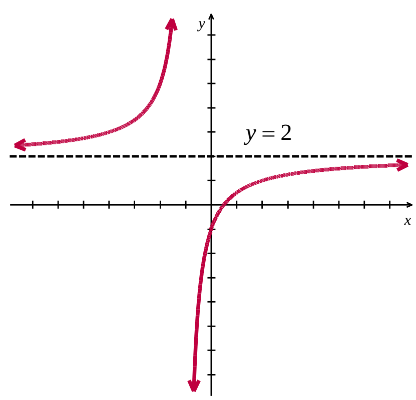
Figure 3.1. 6



End Behavior:

as $x \to -\infty$, $f(x) \to 2$; as $x \to \infty$, $f(x) \to 2$

Figure 3.1. 7



$y=f(x)=\dfrac{2x-1}{x+1}$

[6] Here, the word 'larger' means larger in absolute value.

Definition 3.3. The line $y=c$ is called a **horizontal asymptote** of the graph of a function $y=f(x)$ if, as $x \to -\infty$ or $x \to \infty$, $f(x) \to c$.

Horizontal asymptotes for rational functions occur when the degree of the numerator is less than, or equal to, the degree of the denominator. If the degree of the numerator is greater than the degree of the denominator, the graph will not have a horizontal asymptote.[7] We would like to develop a way of finding horizontal asymptotes without forming tables, as we did above. To develop a general rule, we use the fact that a polynomial's end behavior is governed by its leading term, as found in **Chapter 2**. Let us consider the following two cases.

- We start with the case where the degree of the numerator is less than the degree of the denominator. Consider $f(x)=\dfrac{4x+2}{x^2+4x-5}$. By plugging in large positive or negative *x*-values, such as $f(100)\approx 0.039$ or $f(-100)\approx -0.041$, it seems that $f(x)\to 0$ as $x\to\pm\infty$. A simple explanation, without plugging in *x*-values, is that for large positive or negative *x*-values, $4x+2$ behaves like $4x$ while x^2+4x-5 behaves like x^2. It follows that $f(x)=\dfrac{4x+2}{x^2+4x-5}$ behaves like $y=\dfrac{4x}{x^2}=\dfrac{4}{x}$ for large positive or negative *x*-values. Since, as $x\to\pm\infty$, $y=\dfrac{4}{x}\to 0$, the graph of $y=f(x)$ has a horizontal asymptote of $y=0$.

- Now consider the case where the degree of the numerator is the same as the degree of the denominator. Earlier we saw that the horizontal asymptote of the graph of $f(x)=\dfrac{2x-1}{x+1}$ is $y=2$. A simple explanation, without plugging in *x*-values, is that for large positive or negative *x*-values, $2x+1$ behaves like $2x$ while $x+1$ behaves like x. Thus, $f(x)=\dfrac{2x-1}{x+1}$ behaves like $y=\dfrac{2x}{x}=2$ for large positive or negative *x*-values, or as $x\to\pm\infty$. Notice that 2 is just the ratio of the leading coefficients of the numerator and denominator.

This idea works in general, when there is a horizontal asymptote, and leads to the following result for identifying horizontal asymptotes.

[7] We will talk more about rational functions without horizontal asymptotes in **Section 3.3**.

Identifying Horizontal Asymptotes

Suppose $r(x)=\dfrac{p(x)}{q(x)}$ is a rational function where p and q are polynomial functions. If the degree of p is less than or equal to the degree of q, then r has a horizontal asymptote that may be determined as follows.

1. If the degree of p is less than the degree of q, then the horizontal asymptote is the line $y=0$.
2. If the degree of p is the same as the degree of q, then the horizontal asymptote is the line $y=\dfrac{\text{leading coefficient of } p}{\text{leading coefficient of } q}$.

Example 3.1.5. Identify the horizontal asymptote of the graph for each of the following rational functions.

1. $f(x)=\dfrac{5x}{x^2+1}$
2. $g(x)=\dfrac{x^2-4}{x+1}$
3. $h(x)=\dfrac{6x^3-3x+1}{5-2x^3}$

Solution.

1. The numerator of $f(x)=\dfrac{5x}{x^2+1}$ is $5x$, which has degree 1. The denominator of $f(x)$ is x^2+1, which has degree 2. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y=0$.

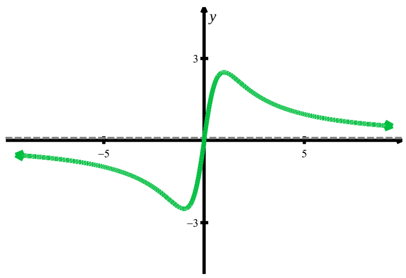
2. The numerator of $g(x)=\dfrac{x^2-4}{x+1}$ is x^2-4, which has degree 2, while the denominator, $x+1$, has degree 1. With the degree of the numerator being greater than the degree of the denominator, we do not have a horizontal asymptote. We will see in **Section 3.3** that the graph of $y=g(x)$ does have what will be referred to as a slant, or oblique, asymptote.

3. The degrees of the numerator and denominator of $h(x)=\dfrac{6x^3-3x+1}{5-2x^3}$ are both three, so the graph of $y=h(x)$ has a horizontal asymptote of $y=\dfrac{6}{-2}$, which simplifies to $y=-3$.

□

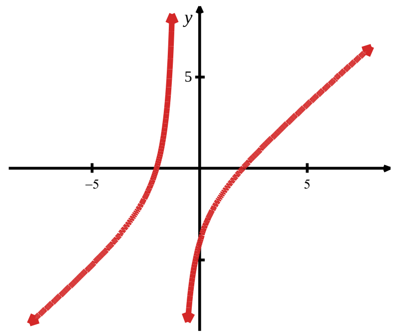
Once again, we employ graphing technology to visualize these results.

Figure 3.1. 8



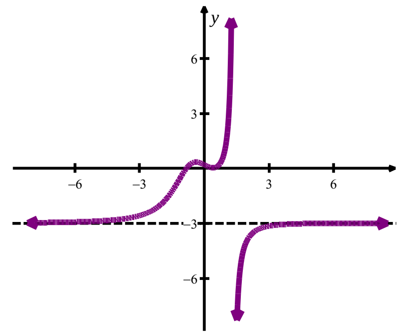
$$y = f(x) = \frac{5x}{x^2+1}$$

Figure 3.1. 9



$$y = g(x) = \frac{x^2-4}{x+1}$$

Figure 3.1. 10



$$y = h(x) = \frac{6x^3-3x+1}{5-2x^3}$$

Graphing

In this section, we limit our graphing of rational functions to irreducible rational functions having denominators of degree one, and numerators of degree at most one. A general procedure for graphing rational functions follows.

Steps for Graphing Rational Functions (specific to Section 3.1)

Suppose r is an irreducible rational function with a constant or first degree numerator and a denominator of degree one.

1. Find the domain of r.
2. Find the x- and y-intercepts, if any exist.
3. Find the vertical asymptote.
4. Find the horizontal asymptote.
5. Identify additional points, as needed, to see how the graph approaches the asymptotes.
6. Plot the intercepts and use dashed lines to sketch the asymptotes, plotting additional points if desired. Sketch the graph, using smooth curves that pass through the intercepts and approach the asymptotes.

Example 3.1.6. Graph the rational function $g(x) = \dfrac{1}{x+2}$.

Solution.

1. We first note that the domain of g excludes $x = -2$, and is therefore $(-\infty,-2)\cup(-2,\infty)$.

2. Since $y = g(x) = \dfrac{1}{x+2} = 0$ has no real solution, there are no *x*-intercepts. Setting $x = 0$ results in $y = \dfrac{1}{0+2}$, for the *y*-intercept $\left(0,\ \dfrac{1}{2}\right)$.

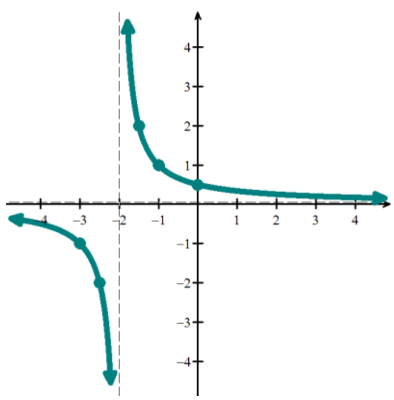
3. Setting the single factor in the denominator equal to zero, we find a vertical asymptote of $x = -2$.

4. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.

5. To see how the graph approaches the asymptotes, we plug in *x*-values near the vertical asymptote, $x = -2$. Choosing the *x*-values -3, -2.5, -1.5, and -1 gives us the corresponding points: $(-3, -1)$, $(-2.5, -2)$, $(-1.5,\ 2)$, and $(-1,\ 1)$.

6. After plotting the *y*-intercept and marking the asymptotes with dashed lines, we plot the additional points from step 5 and sketch a smooth curve, passing through all points and approaching both asymptotes.

Figure 3.1. 11



$$y = g(x) = \frac{1}{x+2}$$

Note that we could have skipped plotting additional points to the right of $x = -2$ since the *y*-intercept is above the *x*-axis and the curve has no *x*-intercepts, so cannot cross the *x*-axis. Additionally, plotting a single point to the left of the vertical asymptote would have been sufficient since, again, there are no *x*-intercepts.

□

We next graph a function similar to $g(x)$, but with both the numerator and the denominator having degree one.

Example 3.1.7. Graph the rational function $h(x) = \dfrac{x+1}{x+2}$.

Solution.

1. Like $g(x)$ in **Example 3.1.6**, the domain of $h(x)=\frac{x+1}{x+2}$ is $(-\infty,-2)\cup(-2,\infty)$.

2. Setting $y=h(x)=0$ gives us $\frac{x+1}{x+2}=0$, from which we have $x+1=0$, resulting in an *x*-intercept of $(-1,0)$. To find the *y*-intercept, we set $x=0$ to get $y=h(0)=\frac{0+1}{0+2}$; the *y*-intercept is $\left(0,\ \frac{1}{2}\right)$.

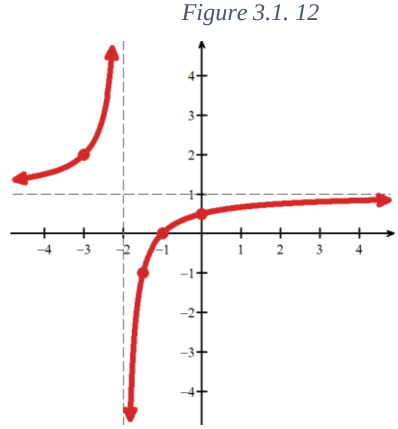
3. After setting the denominator equal to zero, we find that h has a vertical asymptote of $x=-2$.

4. The degrees of the numerator and denominator are the same, so the horizontal asymptote is the line $y=\frac{1}{1}$, or $y=1$.

5. We determine how the graph approaches the asymptotes by plotting additional points. Since there are no *x*-intercepts to the left of $x=-2$, plotting a single point is sufficient. For $x=-3$, we get the point $(-3,\ 2)$. To the right of $x=-2$, there is an *x*-intercept of $(-1,0)$. The *y*-intercept at $\left(0,\ \frac{1}{2}\right)$ tells us the curve is above the *x*-axis to the right of the *x*-intercept. We find an additional point by plugging in $x=-1.5$, which results in the point $(-1.5,-1)$, to determine the curve's behavior between the vertical asymptote and the *x*-intercept.

6. After plotting intercepts, marking asymptotes with dashed lines, and adding additional points, we sketch the graph of $y=h(x)$ with smooth curves that pass through points and approach asymptotes.

Figure 3.1. 12



$$y=h(x)=\frac{x+1}{x+2}$$

□

3.1 Exercises

1. Can the graph of a rational function have no vertical asymptotes? Explain.

2. Can the graph of a rational function have no *x*-intercepts? Explain.

In Exercises 3 – 6, determine if the given function is a rational function. Explain your reasoning.

3. $f(x)=\dfrac{6}{x}$

4. $f(x)=\sqrt{x}-6$

5. $f(x)=\dfrac{1-3x^2}{4x^{\pi}+x^2-1}$

6. $f(x)=\dfrac{2x^6-30}{x^2+5x+3}$

In Exercises 7 – 12, find the domain of the rational function.

7. $f(x)=\dfrac{x-1}{x+2}$

8. $f(x)=\dfrac{x+1}{x^2-1}$

9. $f(x)=\dfrac{x^2+4}{x^2-2x-8}$

10. $f(x)=\dfrac{x^2+4x-3}{x^4-5x^2+4}$

11. $f(x)=\dfrac{x+1}{x^2+25}$

12. $f(x)=\dfrac{x+5}{x^2+25}$

In Exercises 13 – 18, find the *x*- and *y*-intercepts for the rational function.

13. $f(x)=\dfrac{x+5}{x^2+4}$

14. $f(x)=\dfrac{x}{x^2-x}$

15. $f(x)=\dfrac{x^2+8x+7}{x^2+11x+30}$

16. $f(x)=\dfrac{x^2+x+6}{x^2-10x+24}$

17. $f(x)=\dfrac{94-2x^2}{3x^2-12}$

18. $f(x)=\dfrac{x+5}{x^2-25}$

In Exercises 19 – 24, state the vertical asymptote(s). Then determine the end behavior for the rational function and give your answer by stating the following and filling in the blanks.

As $x\to-\infty$, $f(x)\to$ ________, and as $x\to\infty$, $f(x)\to$ ________.

19. $f(x)=\dfrac{4}{x-1}$

20. $f(x)=\dfrac{x-4}{x-6}$

21. $f(x)=\dfrac{x}{x^2-9}$

22. $f(x)=\dfrac{x}{x^2+5x-36}$

23. $f(x)=\dfrac{3x^2+2}{4x^2-1}$

24. $f(x)=\dfrac{3x-4}{x^3-16x}$

In Exercises 25 – 30, identify and state the vertical and horizontal asymptotes for the rational function.

25. $f(x)=\dfrac{x^2-1}{x^3+9x^2+14x}$

26. $f(x)=\dfrac{x+5}{x^2+25}$

27. $f(x)=\dfrac{2}{5x+2}$

28. $f(x)=\dfrac{3+x}{x^3-27}$

29. $f(x)=\dfrac{4-2x}{3x-1}$

30. $f(x)=\dfrac{2x^2-8}{2x^2-4x+2}$

In Exercises 31 – 38, state the domain, intercepts, and asymptotes. Graph the rational function, drawing asymptotes as dashed lines.

31. $f(x)=\dfrac{1}{x-2}$

32. $f(x)=\dfrac{4}{x+2}$

33. $f(x)=\dfrac{2x-3}{x+4}$

34. $f(x)=\dfrac{x-5}{3x-1}$

35. $f(x)=\dfrac{x-3}{x}$

36. $f(x)=\dfrac{5x}{6-2x}$

37. $f(x)=\dfrac{x}{3x-6}$

38. $f(x)=\dfrac{3+7x}{5-2x}$

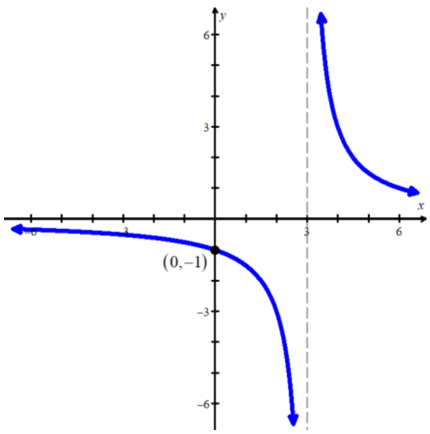
39. Find an equation for a rational function, $r(x)$, that has an *x*-intercept $(1,0)$, a vertical asymptote $x=4$, and a horizontal asymptote $y=3$. The degree of both the numerator and the denominator of $r(x)$ is 1.

40. Find an equation for a rational function, $r(x)$, that has an *x*-intercept $(-2,0)$, a vertical asymptote $x=0$, and a horizontal asymptote $y=-\dfrac{3}{7}$. The degree of both the numerator and the denominator of $r(x)$ is 1.

In Exercises 41 – 44, find an equation for a rational function, $r(x)$, that has the given graph. Write your function in the form $r(x)=\dfrac{ax+b}{cx+d}$.

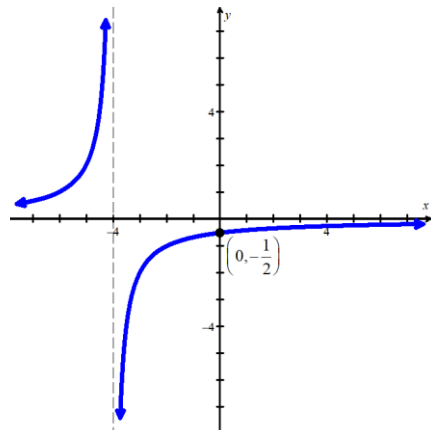
41.

Figure Ex3.1. 1



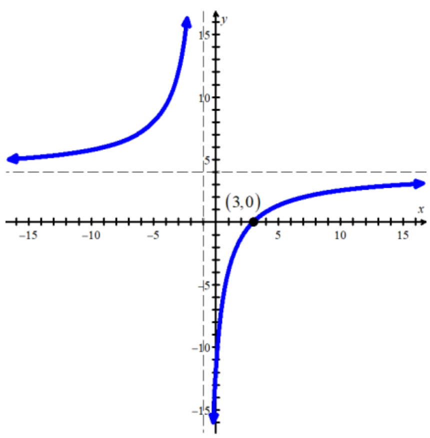
42.

Figure Ex3.1. 2



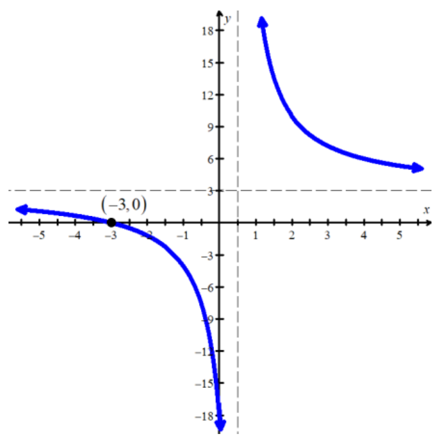
43.

Figure Ex3.1. 3



44.

Figure Ex3.1. 4



45. The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by $C(p)=\dfrac{1770p}{100-p}$, $0\le p<100$.

(a) Find and interpret $C(25)$ and $C(95)$.

(b) What does the vertical asymptote at $x=100$ mean within the context of the problem?

(c) What percentage of the Ippizuti fish can you remove for $40,000?

46. The population of Sasquatch in Portage County can be modeled by the function $P(t)=\dfrac{150t}{t+15}$ where $t=0$ represents the year 1803. Find the horizontal asymptote of the graph of $y=P(t)$ and explain what it means.

3.2 Graphing Rational Functions

Learning Objectives

- Graph irreducible rational functions with denominators of degree greater than one and numerators having the same or a lesser degree.

In this section, we continue graphing rational functions by focusing on functions that have a denominator of degree greater than one. We limit our functions to those with a numerator that has the same degree, or a smaller degree, than the denominator. We follow the general procedure outlined in **Section 3.1**, noting that there may be more than one vertical asymptote, due to having a denominator of degree two or higher. While it would also be possible to have no vertical asymptotes, that particular scenario will be addressed in **Section 3.3**.

Example 3.2.1. Graph the rational function $f(x)=\dfrac{3x}{x^2-4}$.

Solution.

1. To determine the domain of $f(x)=\dfrac{3x}{x^2-4}$, we set the denominator equal to zero to get $x^2-4=0$. Solving this equation, we find that $x=\pm 2$ must be excluded from the domain, so the domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$. Since factoring $f(x)$ results in $f(x)=\dfrac{3x}{(x-2)(x+2)}$, no cancellation of common factors is possible, and we note that f is in lowest terms before moving on.

2. To find the *x*-intercepts of the graph of $y=f(x)$, we consider $y=f(x)=0$. Solving $\dfrac{3x}{(x-2)(x+2)}=0$ results in $3x=0$, from which $x=0$. Since $x=0$ is in the domain of $f(x)$, $(0,0)$ is the *x*-intercept. To find the *y*-intercept, we set $x=0$ and find $y=f(0)=0$ so that $(0,0)$ is the *y*-intercept as well.[8]

3. We find the vertical asymptotes by setting the denominator of $f(x)=\dfrac{3x}{(x-2)(x+2)}$ equal to zero to get $x=\pm 2$. The vertical asymptotes are the lines $x=-2$ and $x=2$.

[8] Since functions can have at most one *y*-intercept, once we find that $(0,0)$ is on the graph, we know it is the *y*-intercept.

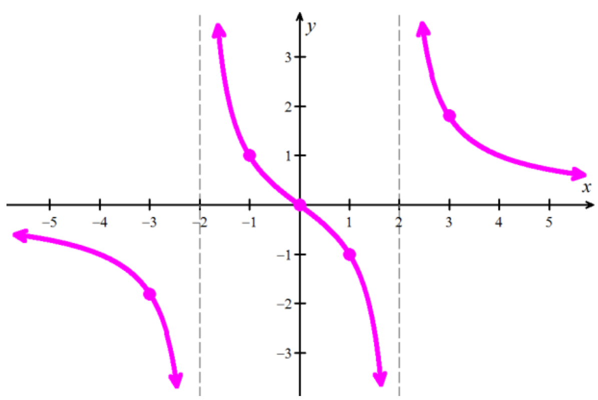
4. We next identify the horizontal asymptote. The degree of the numerator of $f(x)=\frac{3x}{x^2-4}$ is 1 and the degree of the denominator is 2. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y=0$.

5. Finding additional points helps to determine how the graph approaches the asymptotes. Looking for intervals where the graph is above or below the *x*-axis, we note that function values may only change sign at vertical asymptotes and *x*-intercepts. Thus, we choose points in intervals separated by the vertical asymptotes, $x=-2$ and $x=2$, and the *x*-intercept, $(0,0)$.

x	-3	-1	1	3
$f(x)$	-1.8	1	-1	1.8

6. To graph $y=f(x)$, we mark the vertical asymptotes with dashed lines, note that the horizontal asymptote is the *x*-axis, and plot the intercept, along with additional points. We sketch the graph with smooth curves that pass through the intercept and approach the asymptotes, using additional points as guides.

Figure 3.2. 1



$$y=f(x)=\frac{3x}{x^2-4}$$

□

A few notes are in order.

- First, the vertical asymptotes, $x=-2$ and $x=2$, result from zeros in the denominator of $f(x)=\frac{3x}{(x-2)(x+2)}$. Each of these zeros is of multiplicity one and, due to this odd multiplicity, the function changes sign at $x=-2$ and at $x=2$. Making this observation in advance would allow us to plot fewer points when determining the graph's behavior near asymptotes.

- Next, the graph of $y = f(x)$ certainly seems to possess symmetry with respect to the origin. In fact, we can check that $f(-x) = -f(x)$ to see that f is an odd function.
- We see that the graph of f crosses the *x*-axis at $(0,0)$, thus crossing the horizontal asymptote of $y = 0$. While the graph of a rational function may cross its horizontal asymptote, the graph of a rational function will never cross its vertical asymptote. We will see another case of a graph crossing its horizontal asymptote in the next example.

Example 3.2.2. Graph the rational function $g(x) = \dfrac{2-x}{x^3 - 6x^2 + 9x}$.

Solution.

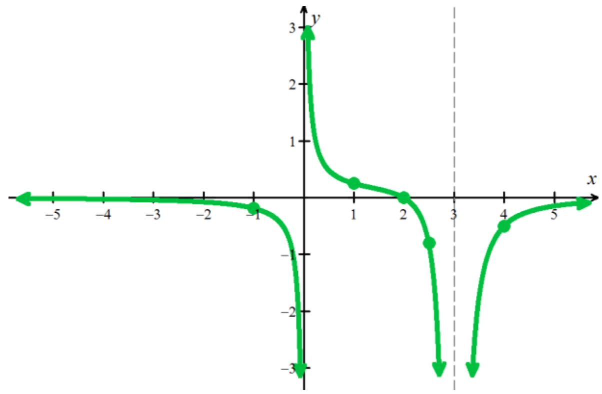
1. To determine the domain, we solve $x^3 - 6x^2 + 9x = 0$ by factoring to get $x(x-3)^2 = 0$, from which $x = 0$ and $x = 3$. The domain is $(-\infty,0)\cup(0,3)\cup(3,\infty)$. Note that $g(x) = \dfrac{2-x}{x(x-3)^2}$ cannot be reduced.
2. To find *x*-intercepts, we solve $y = g(x) = 0$, which happens only if $2 - x = 0$. The solution, $x = 2$, is in the domain of $g(x)$, so the *x*-intercept is $(2,0)$. The function $g(x)$ is undefined when $x = 0$, so there is no *y*-intercept.
3. Setting the denominator of $g(x) = \dfrac{2-x}{x(x-3)^2}$ equal to zero, the vertical asymptotes are $x = 0$ and $x = 3$.
4. Since the degree of the numerator of $g(x) = \dfrac{2-x}{x^3 - 6x^2 + 9x}$ is 1 and the degree of the denominator is 3, the degree of the numerator is less than the degree of the denominator. Thus, the horizontal asymptote is $y = 0$.
5. We look for additional points in intervals separated by the vertical asymptotes, $x = 0$ and $x = 3$, and the *x*-intercept, $(2,0)$.

x	-1	1	2.5	4
$g(x)$	-0.1875	0.25	-0.8	-0.5

6. To graph $y = g(x)$, we note that the *y*-axis, $x = 0$, is a vertical asymptote; we draw the vertical asymptote $x = 3$ with a dashed line; we also note that the *x*-axis, $y = 0$, is the horizontal asymptote.

After plotting the intercept and additional points, we draw smooth curves that pass through these points, while approaching the asymptotes.

Figure 3.2. 2



$$y = g(x) = \frac{2-x}{x^3 - 6x^2 + 9x}$$

□

Before moving on, we look at the denominator of $g(x) = \dfrac{2-x}{x(x-3)^2}$ to see that the vertical asymptote $x = 0$ corresponds to a zero of odd multiplicity one. As expected, the function values change in sign across $x = 0$. The vertical asymptote $x = 3$ corresponds to a zero of even multiplicity two. The function values do not change in sign across $x = 3$, as is the case with a vertical asymptote corresponding to a zero of even multiplicity.

Example 3.2.3. Graph the rational function $h(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$.

Solution.

1. To determine the domain of $h(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$, we solve $x^2 - x - 6 = 0$ to get $x = -2$ and $x = 3$. It follows that the domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$. Factoring the numerator and denominator of $h(x)$ results in $h(x) = \dfrac{(2x-5)(x+1)}{(x-3)(x+2)}$. After observing that no cancellation of common factors is possible, we note that $h(x)$ is in lowest terms.
2. To locate *x*-intercepts, we solve $y = h(x) = 0$. Using the factored form of $h(x)$ above, we find the zeros to be the solutions of $(2x-5)(x+1) = 0$, which are $x = \dfrac{5}{2}$ and $x = -1$. Noting that both of

these numbers are in the domain of h, we have the two x-intercepts $\left(\frac{5}{2},0\right)$ and $(-1,0)$. To find the y-intercept, we input $x=0$ and find $y=h(0)=\frac{5}{6}$, so the y-intercept is $\left(0,\ \frac{5}{6}\right)$.

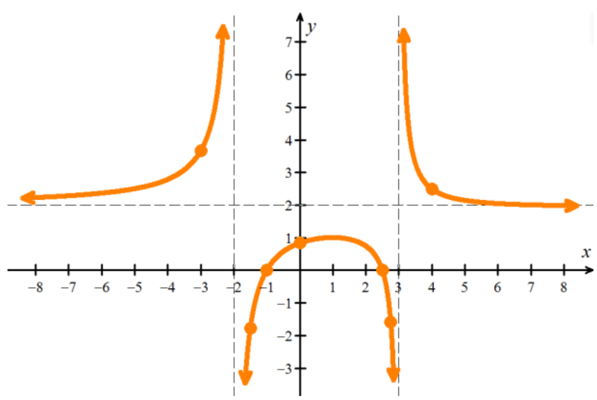
3. To identify vertical asymptotes, we look for values of x that cause the denominator of $h(x)=\frac{(2x-5)(x+1)}{(x-3)(x+2)}$ to be zero. The resulting vertical asymptotes are $x=-2$ and $x=3$.
4. With the numerator and denominator both having degree 2, we use the leading coefficient of each to determine that the horizontal asymptote of the graph of $h(x)=\frac{2x^2-3x-5}{x^2-x-6}$ is $y=\frac{2}{1}=2$.
5. We find additional points in intervals separated by the x-intercepts and vertical asymptotes.

x	-3	-1.5	0	2.75	4
$h(x)$	≈ 3.667	≈ -1.778	≈ 0.833	≈ -1.579	2.5

Noting that the point $\left(0,\ \frac{5}{6}\right)$ was already determined in its role as the y-intercept, we also note that fewer points could be plotted by using odd multiplicities of the zeros of the denominator to indicate sign changes at the vertical asymptotes. We will explore this technique at the end of the section.

6. We mark the asymptotes with dashed lines, plot intercepts and additional points, and use smooth curves to sketch $y=h(x)$, passing through intercepts and additional points while approaching asymptotes.

Figure 3.2. 3



$$y=h(x)=\frac{2x^2-3x-5}{x^2-x-6}$$

□

In the next example, we graph a function without plotting additional points.

Example 3.2.4. Graph the rational function $j(x)=\frac{(x+2)^2(x-3)}{(x+1)^2(x-2)}$.

Solution. We begin with steps from the general graphing procedure in **Section 3.1**, after which we diverge into a more intuitive approach.

1. Setting the denominator, $(x+1)^2(x-2)$ equal to zero, we get $x=-1$ and $x=2$, and so the domain is $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$.

2. To find x-intercepts, we consider $j(x)=0$, which occurs only if $(x+2)^2(x-3)=0$. The solutions of this equation, $x=-2$ and $x=3$, are in the domain of $j(x)$ so the x-intercepts are $(-2,0)$ and $(3,0)$. Inputting $x=0$ gives us $y=j(0)=\frac{(0+2)^2(0-3)}{(0+1)^2(0-2)}=\frac{-12}{-2}$, for the y-intercept of $(0,6)$.

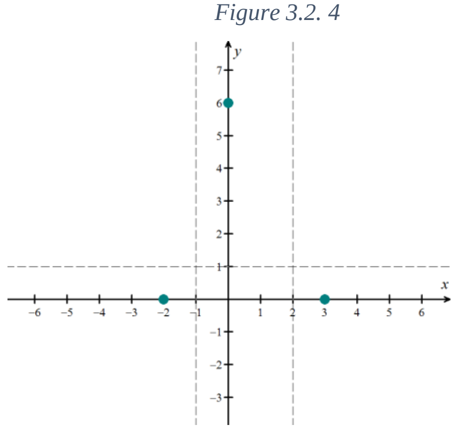
3. From the denominator, we find the vertical asymptotes are $x=-1$ and $x=2$.

4. With the degree of the numerator and denominator being 3, we look at leading coefficients to determine the horizontal asymptote, and find it to be $y=1$.

The remaining steps follow the thought process we might use to graph this function without finding additional points.

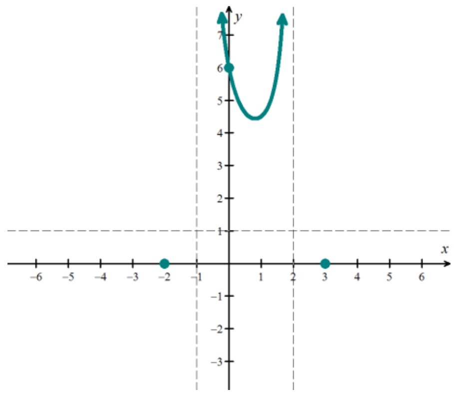
- We plot the information we have so far, including intercepts at $(-2,0)$, $(3,0)$, and $(0,6)$, and asymptotes $x=-1$, $x=2$, and $y=1$.

Figure 3.2. 4



- The graph cannot cross the *x*-axis between the vertical asymptotes since there is no *x*-intercept between $x=-1$ and $x=2$. Since the *y*-intercept is above the *x*-axis, the graph will stay above the *x*-axis, approaching the vertical asymptotes as follows.[9]

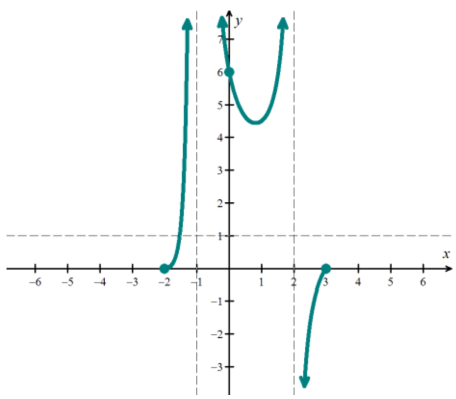
Figure 3.2. 5



While not part of the intuitive process, we can verify our assumptions by finding a couple of points, like $(-0.25,\ 7.8642)$ and $(1.5,\ 5.88)$.

- The vertical asymptote $x=-1$ is a result of the factor $(x+1)^2$ in the denominator. The even power of two tells us the function will not have a sign change at $x=-1$. The vertical asymptote $x=2$ comes from the factor $(x-2)$. The odd power of one indicates a sign change at $x=2$.

Figure 3.2. 6



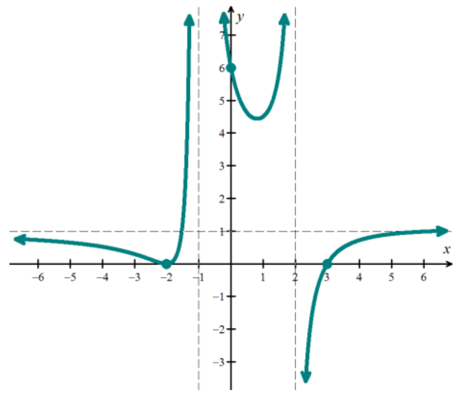
We can again check our results by identifying points; for example $(-1.5,1.29)$ and $(2.5,-1.65)$.[10]

- The *x*-intercept of $(-2,0)$ is a result of $(x+2)^2$ in the numerator. As we discussed when graphing polynomials, the even power prevents a sign change of our function at $x=-2$, and results in the graph merely touching, not crossing, the *x*-axis at that point. The *x*-intercept of $(3,0)$ is a result of

[9] Without graphing additional points it is not possible to know how far down the graph dips. As we have noted, it is even possible that this graph could cross the horizontal asymptote.
[10] *y*-values are approximate.

the factor $(x-3)$, and the odd power of this factor causes a sign change for the function. Thus, the graph crosses the *x*-axis at $(3,0)$. We complete the graph, drawing smooth curves that include appropriate behavior at intercepts and near asymptotes.

Figure 3.2. 7



$$y = j(x) = \frac{(x+2)^2(x-3)}{(x+1)^2(x-2)}$$

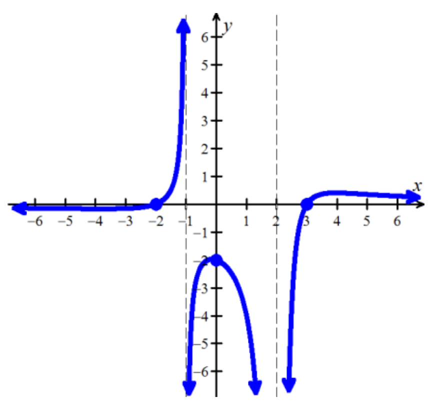
Once again, we can test our graph by calculating points such as $(-3,\ 0.3)$ and $(4,\ 0.72)$.

□

Using intuition to save calculation time when graphing is helpful, although in some cases plotting at least one additional point can prove useful. We end this section with an example in which we determine an equation of a rational function from its graph.

Example 3.2.5. Write an equation for the rational function shown below.

Figure 3.2. 8



Solution. The graph appears to have *x*-intercepts at $x=-2$ and $x=3$. At both points, the graph crosses the *x*-axis, implying zeros with odd multiplicities. For convenience, we will assume multiplicity of one, giving us factors of $(x+2)$ and $(x-3)$ in the numerator.

The graph has two vertical asymptotes. At $x = -1$, the function has a sign change, so we can assume that the corresponding factor has an odd exponent; we choose one for simplicity. There is no sign change at $x = 2$, so the corresponding factor must have an even exponent, and we will assume that multiplicity is two. Thus, we have the factors $(x+1)$ and $(x-2)^2$ in the denominator.

Putting together what we know so far, the function looks like $f(x) = a \cdot \frac{(x+2)(x-3)}{(x+1)(x-2)^2}$ for some constant a that is yet unknown. We do have one additional piece of information, and that is the *y*-intercept at $(0,-2)$. We plug in 0 for x and -2 for $y = f(x)$ to solve for a.

$$-2 = a \cdot \frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$

$$-2 = a \cdot \frac{-6}{4}$$

$$-2 \cdot \frac{4}{-6} = a$$

$$a = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$. As a last check, we confirm that the degree of the numerator is less than the degree of the denominator, verifying that our function has the required horizontal asymptote of $y = 0$.

□

Note that in **Example 3.2.5**, the equation we have found works for a function that has the intercepts, asymptotes, and general behavior indicated by the graph. The solution is not unique if we are only basing it on these characteristics. Other even powers could be used in place of two, and other odd powers could be used in place of one. The reader is encouraged to think about the role of factors and their powers within rational functions before moving on.

3.2 Exercises

1. How can multiplicities be used in graphing rational functions?

2. What property of a rational function results in its graph crossing the x-axis at an intercept?

In Exercises 3 – 14, state the domain, intercepts, and asymptotes. Graph the rational function, drawing asymptotes as dashed lines.

3. $f(x)=\dfrac{1}{x^2}$

4. $f(x)=\dfrac{1}{x^2+x-12}$

5. $f(x)=\dfrac{x}{x^2+x-12}$

6. $f(x)=\dfrac{4x}{x^2-4}$

7. $f(x)=\dfrac{3x^2-5x-2}{x^2-9}$

8. $f(x)=\dfrac{x+7}{x^2+6x+9}$

9. $f(x)=\dfrac{4}{x^2-4x+4}$

10. $f(x)=\dfrac{5}{x^2+2x+1}$

11. $f(x)=\dfrac{3x^2-14x-5}{3x^2+8x-16}$

12. $f(x)=\dfrac{2x^2+7x-15}{3x^2-14x+15}$

13. $f(x)=\dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

14. $f(x)=\dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

In Exercises 15 – 20, write an equation for a rational function with the given characteristics.

15. Vertical asymptotes at $x=5$ and $x=-5$; x-intercepts at $(2,0)$ and $(-1,0)$; y-intercept at $(0,4)$

16. Vertical asymptotes at $x=-4$ and $x=-1$; x-intercepts at $(1,0)$ and $(5,0)$; y-intercept at $(0,7)$

17. Vertical asymptotes at $x=-4$ and $x=-5$; x-intercepts at $(4,0)$ and $(-6,0)$; horizontal asymptote at $y=7$

18. Vertical asymptotes at $x=-3$ and $x=6$; x-intercepts at $(-2,0)$ and $(1,0)$; horizontal asymptote at $y=-2$

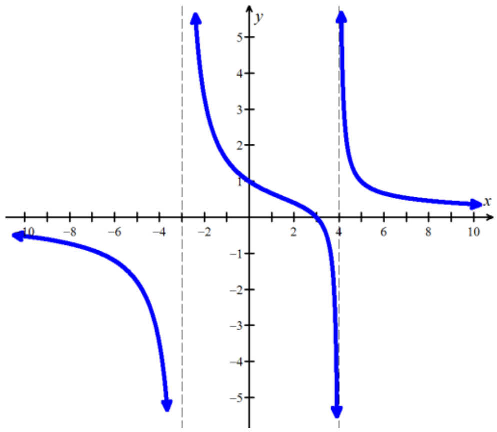
19. Vertical asymptote at $x=-1$; graph touches but does not cross x-axis at $(2,0)$; y-intercept at $(0,2)$

20. Vertical asymptote at $x=3$; graph touches but does not cross x-axis at $(1,0)$; y-intercept at $(0,4)$

In Exercises 21 – 26, use the graph to write an equation for the function.

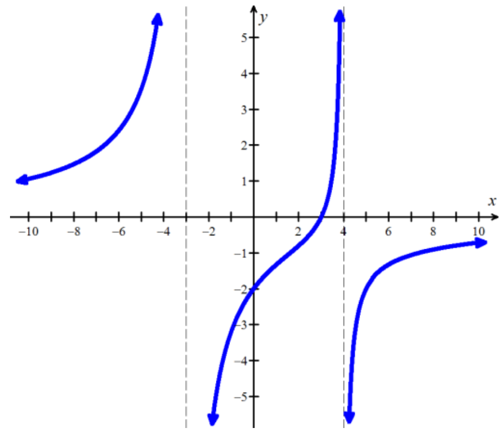
21.

Figure Ex3.2. 1



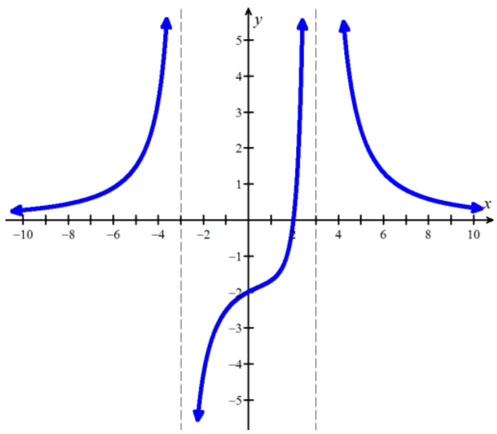
22.

Figure Ex3.2. 2



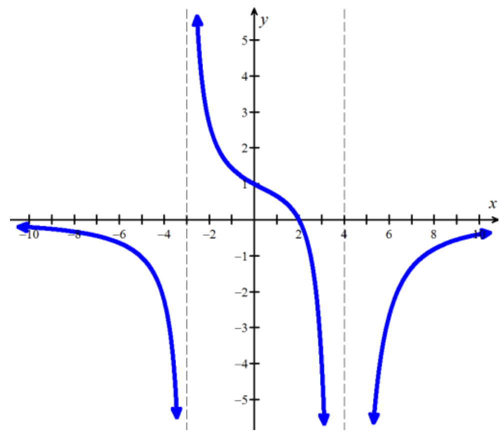
23.

Figure Ex3.2. 3



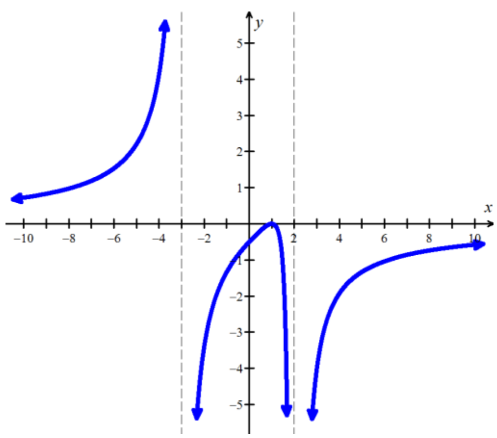
24.

Figure Ex3.2. 4



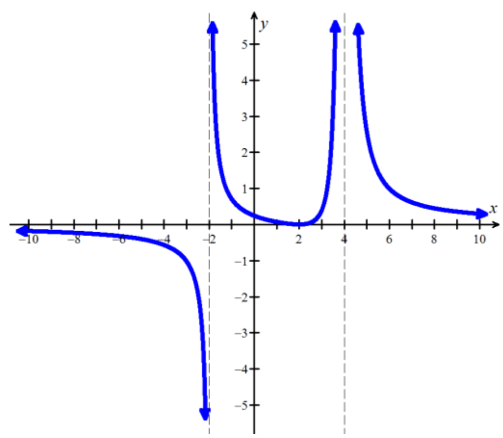
25.

Figure Ex3.2. 5



26.

Figure Ex3.2. 6



3.3 More with Graphing Rational Functions

Learning Objectives

- Identify holes in the graph of a rational function.
- Graph rational functions without vertical asymptotes.
- Find slant (oblique) asymptotes.
- Graph rational functions having slant asymptotes.

In this section, we look at rational functions whose graphs have holes, do not have vertical asymptotes, or have slant asymptotes.

Holes in Graphs of Rational Functions

Graphs of rational functions do not contain any points on their vertical asymptotes since rational functions are not defined for x-values where vertical asymptotes occur. Additionally, it is possible for the graph of a rational function to exclude a point that is not on a vertical asymptote. Such a point is referred to as a **hole** and is indicated on the graph by a small circle centered at that point. The function $H(x)=\dfrac{x+3}{x^2-9}$ is an example of a rational function whose graph contains a hole, and we graph this function in the following example.

Example 3.3.1. Graph the rational function $H(x)=\dfrac{x+3}{x^2-9}$.

Solution. We follow a procedure similar to that in **Section 3.1**, adding to the steps as necessary to account for potential holes.

1. The values $x=\pm 3$ result in a denominator of zero, so our domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

   We proceed with factoring before reducing the function to lowest terms.

$$\begin{aligned} H(x) &= \frac{x+3}{(x-3)(x+3)} \\ &= \frac{1}{x-3},\ x\neq -3 \end{aligned}$$

   Since $x=-3$ is not in the domain of H, it must be excluded as a value for x in the simplified version of $H(x)$, and a point having an x-coordinate of -3 cannot be included in the graph of the function. We will indicate this excluded point on the graph with a small circle centered at that point.

This means that the graph of $y = H(x)$ is the graph of $y = K(x) = \dfrac{1}{x-3}$ with the point $\left(-3, \dfrac{1}{-3-3}\right) = \left(-3, -\dfrac{1}{6}\right)$ removed. So, from here we proceed to graph $K(x) = \dfrac{1}{x-3}$.

2. We next locate the intercepts. We observe that there are no *x*-intercepts since $\dfrac{1}{x-3} \neq 0$ for any value of x. There is a *y*-intercept when $y = \dfrac{1}{0-3}$, at the point $\left(0, -\dfrac{1}{3}\right)$.

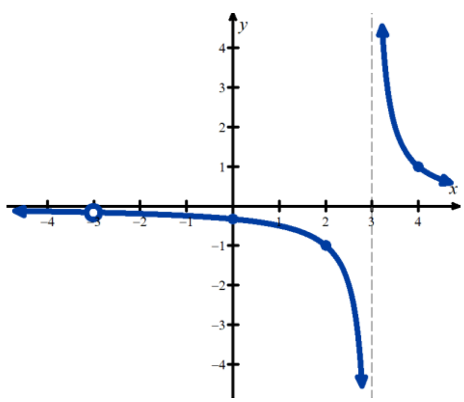
3. The vertical asymptote for the graph of $y = K(x)$, or $y = H(x)$, is the line $x = 3$. Notice that $x = -3$ is not a vertical asymptote since the factor $(x+3)$ was canceled from the denominator.

4. The degree of the numerator is less than the degree of the denominator, so the horizontal asymptote is $y = 0$.

5. Two additional points on the graph are $(2, -1)$ and $(4, 1)$. The point we should remove is $\left(-3, -\dfrac{1}{6}\right)$.

6. Using the above information, we sketch the graph.

Figure 3.3. 1



$$y = H(x) = \frac{x+3}{x^2-9}$$

□

Graphing Rational Functions without Vertical Asymptotes

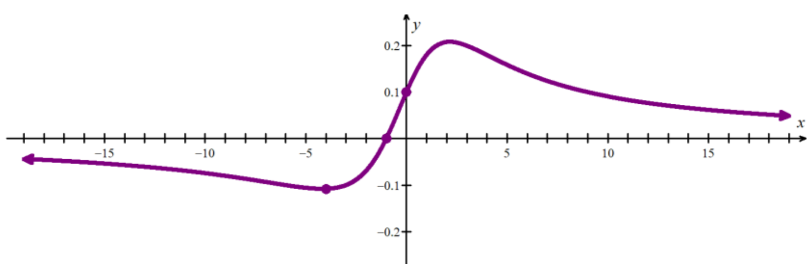
The domain of the function $G(x) = \dfrac{x+1}{x^2+9}$ is the set of all real numbers and its graph does not have a vertical asymptote since its denominator $x^2 + 9$ is never zero.

Example 3.3.2. Graph the rational function $G(x) = \dfrac{x+1}{x^2+9}$.

Solution.

1. The denominator x^2+9 is never zero, so the domain of G is $(-\infty,\infty)$. No factoring or further simplification of G is possible.
2. To find x-intercepts, consider $G(x)=0$, which occurs only if $x+1=0$, resulting in an x-intercept of $(-1,0)$. Inputting $x=0$ to $G(x)=\dfrac{x+1}{x^2+9}$ gives us the y-intercept of $\left(0,\ \dfrac{1}{9}\right)$.
3. Since no values of x cause the denominator to be zero, there are no vertical asymptotes.
4. With the degree of the numerator being 1 and the degree of the denominator being 2, the degree of the numerator is less that the degree of the denominator and so the horizontal asymptote is $y=0$.
5. There are no vertical asymptotes, only the x-intercept of $(-1,0)$ that might separate positive from negative function values. To the right of $(-1,0)$, the y-intercept of $\left(0,\ \dfrac{1}{9}\right)$ tells us the graph is above the x-axis. To the left, we find the point $\left(-4,-\dfrac{3}{25}\right)$.
6. The horizontal asymptote, $y=0$, is the x-axis. We plot the intercepts and the additional point, then sketch the graph of $y=G(x)$ with a smooth curve that approaches the horizontal asymptote as $x\to-\infty$ and as $x\to\infty$.

Figure 3.3. 2



$$y=G(x)=\frac{x+1}{x^2+9}$$

□

Identifying Slant Asymptotes

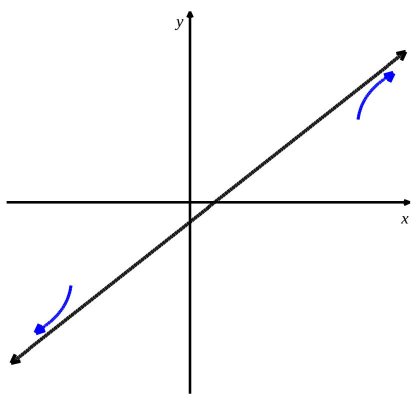
We finish this section, and our graphing of rational functions, with the third (and final!) type of asymptote that can be associated with the graphs of rational functions. For the function $f(x)=\dfrac{x^2-4}{x+1}$, if we perform long division on $\dfrac{x^2-4}{x+1}$, we get $f(x)=x-1-\dfrac{3}{x+1}$. Since the term $\dfrac{3}{x+1}\to 0$ as $x\to\pm\infty$, it stands to reason that for large positive or negative *x*-values, the function values are close to that of $y = x - 1$. That is, $f(x) = x - 1 - \frac{3}{x+1} \approx x - 1$ for large positive or negative *x*-values. Geometrically, this means that the graph of $y = f(x)$ should be close to the line $y = x - 1$ as $x \to \pm\infty$. We see this play out numerically and graphically below.

x	$f(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

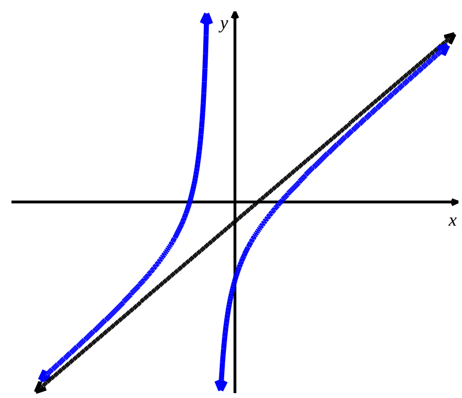
Figure 3.3. 3

x	$f(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

Figure 3.3. 4



$y = x - 1$ and behavior of $f(x)$ as $x \to \pm\infty$



$y = x - 1$ and $y = f(x)$

In this case, we say the line $y = x - 1$ is a **slant asymptote**[11] to the graph of $y = f(x)$. Informally, we say the nonhorizontal (slant) line $y = x - 1$ is the slant asymptote of the graph of $y = f(x)$ if, as $x \to -\infty$ or as $x \to \infty$, $f(x) \approx x - 1$, or the graph of $y = f(x)$ resembles that of $y = x - 1$. Formally, this can be stated as: $y = x - 1$ is the slant asymptote of the graph of $y = f(x)$ if, as $x \to -\infty$ or as $x \to \infty$, $\left[f(x) - (x-1)\right] \to 0$.

Definition 3.4. The line $y = mx + b$, where $m \neq 0$, is called a **slant asymptote** of the graph of a function $y = f(x)$ if, as $x \to -\infty$ or as $x \to \infty$, $\left[f(x) - (mx+b)\right] \to 0$.

[11] Also called an **oblique asymptote** in some, ostensibly higher class (and more expensive), texts.

This means that if the line $y = mx + b$, where $m \neq 0$, is a slant asymptote of the graph of a function $y = f(x)$, then, as $x \to -\infty$ or as $x \to \infty$, $f(x) \approx mx + b$ (the graph of $y = f(x)$ resembles that of $y = mx + b$).

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote and, if it does, how to find it. In the case of $f(x) = \frac{x^2 - 4}{x+1}$, the degree of the numerator $x^2 - 4$ is 2, which is exactly one more than the degree of its denominator $x + 1$, which is 1. This results in a linear quotient polynomial, and it is this quotient polynomial that is the slant asymptote. Generalizing this situation gives us the following theorem.[12]

Theorem 3.1. Determination of Slant Asymptotes: Suppose $r(x) = \frac{p(x)}{q(x)}$ is a rational function where the degree of p is exactly one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

Unlike the shortcut we have been using to find horizontal asymptotes, there is no recourse in finding slant asymptotes but to use long or synthetic division. We will demonstrate this in the first problem of the following example.

Example 3.3.3. Identify the slant asymptote of the graph of the following rational functions, if one exists.

1. $f(x) = \frac{x^2 - 4x + 2}{1 - x}$
2. $g(x) = \frac{x^2 - 4}{x - 2}$
3. $h(x) = \frac{x^3 + 1}{x^2 - 4}$

Solution.

1. For $f(x) = \frac{x^2 - 4x + 2}{1 - x}$, the degree of the numerator is 2 and the degree of the denominator is 1, so **Theorem 3.1** guarantees us a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$.

$$\begin{array}{r} -x+3 \\ -x+1 \overline{\smash{\big)}\, x^2 - 4x + 2} \\ -\underline{\left(x^2 - x\right)} \quad\quad \\ -3x + 2 \\ -\underline{\left(-3x + 3\right)} \\ -1 \end{array}$$

[12] This theorem is brought to you courtesy of **Theorem 2.3** and Calculus.

The result is a quotient of $-x+3$ with remainder -1. The slant asymptote is given by the quotient of this long division. That is, the slant asymptote is the line $y=-x+3$. Notice that, in this case, unlike for horizontal asymptotes, the ratio of leading terms of the numerator and denominator, $\frac{x^2}{-x}=-x$, does not give the slant asymptote since it is not the quotient of the long division. As stated above, there is no recourse in finding slant asymptotes but to use long division.

2. As with the previous example, the degree of the numerator of $g(x)=\frac{x^2-4}{x-2}$ is 2 and the degree of the denominator is 1, so **Theorem 3.1** applies. In this case,

$$\begin{aligned} g(x) &= \frac{x^2-4}{x-2} \\ &= \frac{(x+2)(x-2)}{(x-2)} \\ &= x+2, \quad x \neq 2 \end{aligned}$$

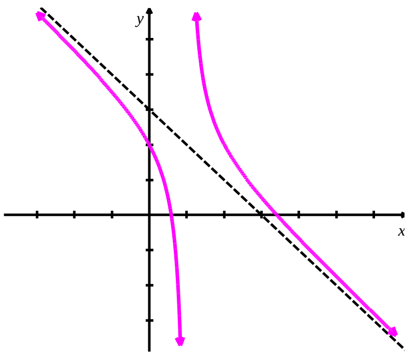
We see that when we divide $x-2$ into x^2-4, we get a quotient of $x+2$ (with remainder 0), and so we have the slant asymptote $y=x+2$. We note that the graph of the slant asymptote is identical to the graph of $y=g(x)$ except at $x=2$, where the latter has a hole at $(2,4)$.

3. For $h(x)=\frac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2; again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3+1\right)\div\left(x^2-4\right)$ gives a quotient of just x, so the slant asymptote is the line $y=x$.

□

The graphs of the three functions from **Example 3.3.3** follow, with slant asymptotes represented by dashed lines. Following these three graphs, we will proceed with our own graphing of rational functions that have slant asymptotes.

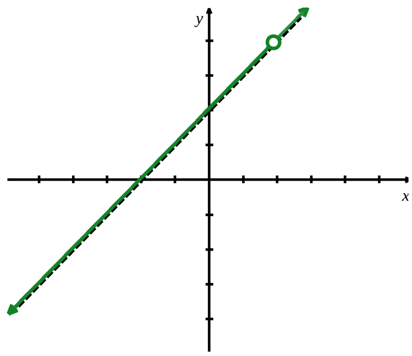
Figure 3.3. 5



$y = f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$

slant asymptote $y = -x + 3$

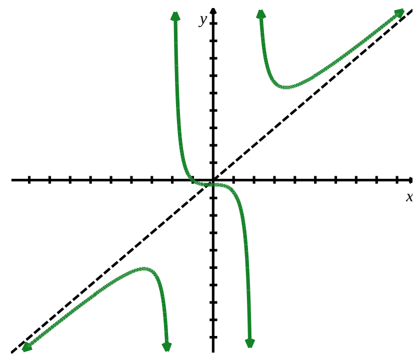
Figure 3.3. 6



$y = g(x) = \dfrac{x^2 - 4}{x - 2}$

slant asymptote $y = x + 2$

Figure 3.3. 7



$y = h(x) = \dfrac{x^3 + 1}{x^2 - 4}$

slant asymptote $y = x$

A rational function may have a horizontal asymptote or a slant asymptote, but not both. We note that the method for finding a slant asymptote also works for finding a horizontal asymptote. In **Section 3.1**, we found that the horizontal asymptote of the graph of $f(x) = \dfrac{2x - 1}{x + 1}$ is $y = 2$. Using long division, we see that $f(x) = 2 - \dfrac{3}{x + 1}$, so 2 is the quotient and, hence, $y = 2$ is the horizontal asymptote. Similarly, $y = 0$ is the horizontal asymptote of the graph of $f(x) = \dfrac{4x + 2}{x^2 + 4x - 5}$ since the numerator has smaller degree than the denominator. Using long division, we find $f(x) = 0 + \dfrac{4x + 2}{x^2 + 4x - 5}$. Since the quotient is zero, the horizontal asymptote is $y = 0$. Although, mathematically, horizontal and slant asymptotes are the same, for ease of calculation we have described different ways of finding them.

Graphing Rational Functions that have Slant Asymptotes

Before moving on to graphing functions with slant asymptotes, we revisit the steps for graphing rational functions, changing the wording a bit to accommodate new information.

Steps for Graphing Rational Functions

Suppose r is a rational function.

1. Find the domain of r. After recording the domain, identify the location of any holes and reduce r to lowest terms, if possible. From this point, proceed with the reduced function but with the domain of r.
2. Find the *x*- and *y*-intercepts, if any exist.
3. Find the vertical asymptotes, if any exist.
4. Find the horizontal or slant asymptote, if one exists.
5. Find additional points, as needed.
6. Plot the intercepts and indicate the holes, use dashed lines to sketch the asymptotes, and add additional points if desired. Sketch the graph, using smooth curves that pass through the intercepts and additional points and approach the asymptotes.

In the following two examples, we demonstrate these steps for graphing rational functions.

Example 3.3.4. Graph the rational function $f(x)=\dfrac{3x^2-2x+1}{x-1}$.

Solution. We follow the steps listed above.

1. Setting the denominator equal to zero, we get $x=1$, for a domain of $(-\infty,1)\cup(1,\infty)$. The numerator is not factorable, so $f(x)$ cannot be written in lower terms, and there are subsequently no holes.
2. Setting $y=f(x)=0$, we look for *x*-intercepts where $3x^2-2x+1=0$. After applying the quadratic formula, we find there are no real solutions, and conclude that there are no *x*-intercepts. To find the *y*-intercept, we have $y=f(0)=\dfrac{3(0)^2-2(0)+1}{0-1}=-1$, for a *y*-intercept of $(0,-1)$.
3. Setting the denominator of $x-1$ equal to zero results in a vertical asymptote of $x=1$.
4. The degree of the numerator is 2 and the degree of the denominator is 1, so we have a slant asymptote. After using long division[13], we have $(3x^2-2x+1)\div(x-1)=3x+1+\dfrac{2}{x-1}$, from which we find that the slant asymptote is $y=3x+1$.

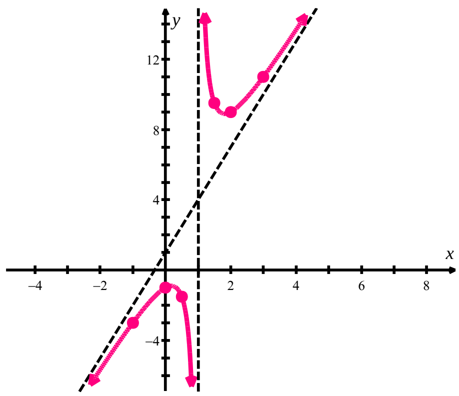
[13] Synthetic division works here.

5. Since there are no x-intercepts, the function can only change sign at the vertical asymptote, $x=1$. Therefore, we find additional points on both sides of $x=1$.

x	-1	0.5	1.5	2	3
$f(x)$	-3	-1.5	9.5	9	11

6. To graph $y=f(x)$, we mark the asymptotes with dashed lines, plot the intercept and extra points, then use smooth curves to draw the graph, passing through the intercept and using the points to guide us in approaching both the vertical and slant asymptote.

Figure 3.3. 8



$$y=f(x)=\frac{3x^2-2x+1}{x-1}$$

□

Example 3.3.5. Draw the graph of the rational function $g(x)=\dfrac{2x^3+5x^2+4x+1}{x^2+3x+2}$.

Solution.

1. To determine the domain, we set the denominator equal to zero to find values of $x=-1$ and $x=-2$. Thus, the domain is $(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$. We next factor the numerator and denominator to determine any holes and then reduce g to simplest terms.

   To factor $2x^3+5x^2+4x+1$, we use the Rational Zeros Theorem to identify potential rational zeros of ± 1 and $\pm\frac{1}{2}$. We then follow with synthetic division that verifies $x=-1$ is a zero.

$$\begin{array}{r|rrrr} -1 & 2 & 5 & 4 & 1 \\ & \downarrow & -2 & -3 & -1 \\ \hline & 2 & 3 & 1 & 0 \end{array}$$

We use the results to factor $g(x)=\frac{(x+1)(2x^2+3x+1)}{x^2+3x+2}=\frac{(x+1)(x+1)(2x+1)}{(x+1)(x+2)}$. After noting that there is a hole in the graph at $x=-1$, we write g as $g(x)=\frac{(x+1)(2x+1)}{x+2}$ but remember that its domain is still $(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$, which does not include $x=-1$.

2. To find *x*-intercepts, we solve $g(x)=\frac{(x+1)(2x+1)}{x+2}=0$ to get $x=-1$ and $x=-\frac{1}{2}$. Since there is a hole in the graph at $x=-1$, it will not correspond to an *x*-intercept, but it will be important to indicate a hole at $(-1,0)$ when sketching the graph. To find the *y*-intercept, we set x equal to zero and find $y=\frac{(0+1)(2\cdot 0+1)}{0+2}=\frac{1}{2}$. Thus, the intercepts are $\left(-\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$.

3. Setting the single remaining factor in the denominator equal to zero, $x+2=0$, yields a vertical asymptote of $x=-2$.

4. In $g(x)=\frac{(x+1)(2x+1)}{x+2}=\frac{2x^2+3x+1}{x+2}$, the degree of the numerator is 2 and the degree of the denominator is 1. We have a slant asymptote since the degree of the numerator is one more than the degree of the denominator. With a divisor of the form $x-c$, we use synthetic division to perform the long division.

$$
\begin{array}{r|rrr}
-2 & 2 & 3 & 1 \\
 & \downarrow & -4 & 2 \\
\hline
 & 2 & -1 & 3
\end{array}
$$

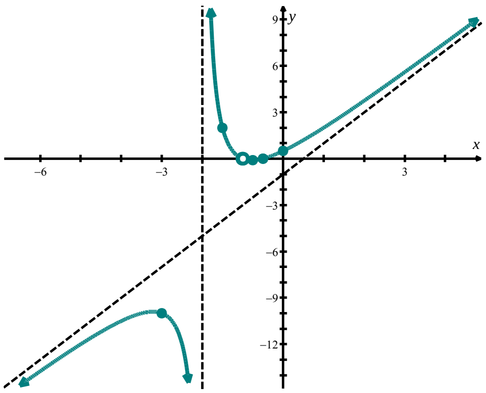
So $(2x^2+3x+1)\div(x+2)=2x-1+\frac{3}{x+2}$ and the slant asymptote is $y=2x-1$.

5. We find additional points in intervals separated by the vertical asymptote, the *x*-intercept, and, in this case, the hole that occurs at the point $(-1,0)$ since it is on the *x*-axis.

x	-3	-1.5	-0.75
$g(x)$	-10	2	-0.1

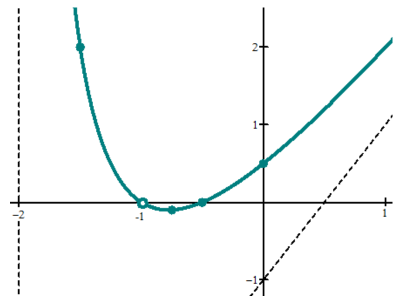
6. After marking the asymptotes with dashed lines and plotting the intercepts, hole, and additional points, we complete the graph with smooth curves that approach both asymptotes.

Figure 3.3. 9



$$y = g(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$$

Figure 3.3. 10



a closer look at the interval $(-2,1)$

□

Notice that in **Figure 3.3.9**, it is difficult to see what is going on between $x = -1$ and $x = -\frac{1}{2}$. **Figure 3.3.10** provides a closer look at the behavior of the graph in that region.

3.3 Exercises

1. What characteristics of a rational function indicate the absence of vertical asymptotes?

2. What characteristics of a rational function indicate the presence of a slant asymptote?

In Exercises 3 – 8, determine if a slant asymptote exists. If a slant asymptote exists, state its equation.

3. $f(x)=\dfrac{24x^2+6x}{2x+1}$

4. $f(x)=\dfrac{4x^2-10}{2x-4}$

5. $r(x)=\dfrac{81x^2-18}{3x-2}$

6. $f(x)=\dfrac{6x^3-5x}{3x^2+4}$

7. $f(x)=\dfrac{x^3}{1-x}$

8. $g(x)=\dfrac{x^2+5x+4}{x-1}$

In Exercises 9 – 12, state the holes in the graph of the rational function as ordered pairs. Write the rational function in its reduced form with the domain restriction.

9. $f(x)=\dfrac{x^2+2x-3}{x^2-1}$

10. $r(x)=\dfrac{x^2-x-6}{x^2-4}$

11. $r(x)=\dfrac{2x-1}{-2x^2-5x+3}$

12. $f(x)=\dfrac{x^2-x-12}{x^2+x-6}$

In Exercises 13 – 32, state the domain, location of any holes (written as ordered pairs), intercepts, and asymptotes. Graph the rational function, drawing asymptotes as dashed lines.

13. $f(x)=\dfrac{x^2+2x-3}{x^2-1}$

14. $r(x)=\dfrac{x^2-x-6}{x^2-4}$

15. $f(x)=\dfrac{2x^2+x-1}{x-4}$

16. $g(x)=\dfrac{2x^2-3x-20}{x-5}$

17. $r(x)=\dfrac{2x-1}{-2x^2-5x+3}$

18. $r(x)=\dfrac{4x}{x^2+4}$

19. $g(x)=\dfrac{x^2-x-12}{x^2+x-6}$

20. $f(x)=\dfrac{x^2-x-6}{x+1}$

21. $f(x)=\dfrac{x^2-x}{3-x}$

22. $h(x)=\dfrac{x^3+2x^2+x}{x^2-x-2}$

23. $r(x) = \dfrac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$

24. $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$

25. $r(x) = \dfrac{x - 2}{x^2 - 4}$

26. $f(x) = \dfrac{x^2 - 25}{x^3 - 6x^2 + 5x}$

27. $f(x) = \dfrac{x^3 + 1}{x^2 - 1}$

28. $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$

29. $r(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$

30. $g(x) = \dfrac{18 - 2x^2}{x^2 - 9}$

31. $f(x) = \dfrac{2x^3 - x^2 - 3x}{x^3 - 6x^2 + 9x}$

32. $r(x) = \dfrac{3(x-2)^2(x+3)}{(2x-3)^2(x+1)^3}$

33. The six-step graphing procedure outlined in **Section 3.3** cannot tell us everything of importance about the graph of a rational function. Without Calculus, we may use graphing technology to reveal the hidden mysteries of rational function behavior. Working with your classmates, use graphing technology to examine the graphs of the following rational functions. Compare and contrast their features. Which of the features can the six-step process reveal and which features cannot be detected by it?

(a) $f(x) = \dfrac{1}{x^2 + 1}$

(b) $g(x) = \dfrac{x}{x^2 + 1}$

(c) $h(x) = \dfrac{x^2}{x^2 + 1}$

(d) $r(x) = \dfrac{x^3}{x^2 + 1}$

3.4 Solving Rational Equations and Inequalities

Learning Objectives

- Solve rational equations.
- Solve rational inequalities graphically.
- Solve rational inequalities algebraically.

In this section, we solve equations and inequalities involving rational functions. We begin with rational equations, which you are likely familiar with from prior mathematics classes.

Solving Rational Equations

As with polynomial equations, we use graphs and algebra to solve rational equations.

Example 3.4.1. Solve each of the following equations, graphically and algebraically.

1. $\frac{2x-1}{x+3}=0$
2. $\frac{2x-1}{x+3}=4$

Solution. We begin determining a graphical solution for each equation, letting $f(x)=\frac{2x-1}{x+3}$, $g(x)=0$, and $h(x)=4$. The solution to $\frac{2x-1}{x+3}=0$ is found where $f(x)=g(x)$ and the solution to $\frac{2x-1}{x+3}=4$ occurs when $f(x)=h(x)$.

Figure 3.4. 1

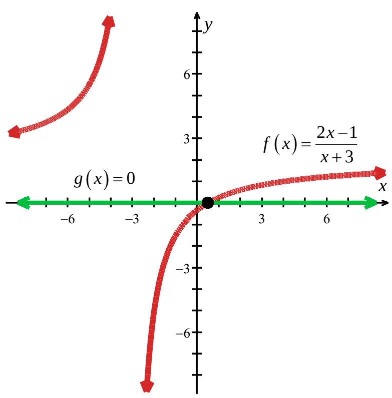
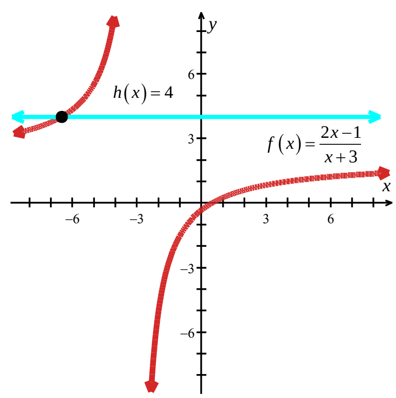


Figure 3.4. 2



While the points of intersection are not easily identifiable from these graphs, we will find through algebra that $f(x)=g(x)$ at the point $\left(\frac{1}{2},0\right)$ and that $f(x)=h(x)$ at the point $\left(-\frac{13}{2},4\right)$. Thus, the solution to $\frac{2x-1}{x+3}=0$ is $x=\frac{1}{2}$ and the solution to $\frac{2x-1}{x+3}=4$ is $x=-\frac{13}{2}$.

To solve the equations algebraically, we see that $\frac{2x-1}{x+3}=0$ when $2x-1=0$, as long as $x+3\neq 0$, so the solution is $x=\frac{1}{2}$. For $\frac{2x-1}{x+3}=4$, we have

$$\begin{aligned}\frac{2x-1}{x+3}&=4\\ 2x-1&=4(x+3)\\ 2x-1&=4x+12\\ -2x&=13\\ x&=-\frac{13}{2}\end{aligned}$$

The solution is $x=-\frac{13}{2}$ after checking that, for this value, the denominator $x+3$ is not zero.

□

A few observations are in order before moving on.

- The solution to $\frac{2x-1}{x+3}=0$ is the same as the *x*-coordinate of the *x*-intercept for the graph of $f(x)=\frac{2x-1}{x+3}$.
- The denominator of the function from the previous example, $f(x)=\frac{2x-1}{x+3}$, gives us a domain restriction when $x+3=0$; this is the location of the vertical asymptote $x=-3$.
- Each equation from the previous example has a single solution, as indicated by their graphs.

Example 3.4.2. Solve $\frac{x+1}{x^2-1}=0$.

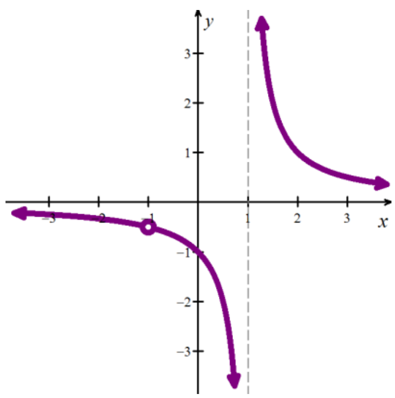
Solution. Here, we begin by solving the equation algebraically, factoring as follows.

$$\begin{aligned}\frac{x+1}{x^2-1}&=0\\ \frac{x+1}{(x-1)(x+1)}&=0\end{aligned}$$

While the only possible solution occurs when the numerator is zero, in this case that potential solution of $x=-1$ is not allowed since this value makes the denominator zero. Thus, the equation $\frac{x+1}{x^2-1}=0$ does not have any solutions.

For a graphical solution, we graph $y=\frac{x+1}{x^2-1}=\frac{1}{x-1}$, if $x\neq -1$. Note that the graph of $y=\frac{x+1}{x^2-1}$ (seen below) is the same as the graph of $y=\frac{1}{x-1}$, except for a hole at the point where $x=-1$. Due to the absence of *x*-intercepts, this function is never equal to zero, confirming the conclusion that there is no solution.

Figure 3.4. 3



$$y=\frac{x+1}{x^2-1}$$

□

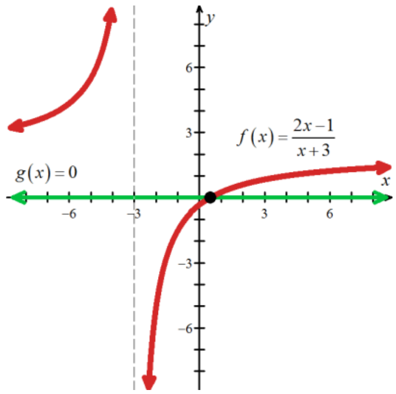
We move on to solving rational inequalities.

Solving Rational Inequalities Graphically

Example 3.4.3. Solve $\frac{2x-1}{x+3}\geq 0$ graphically.

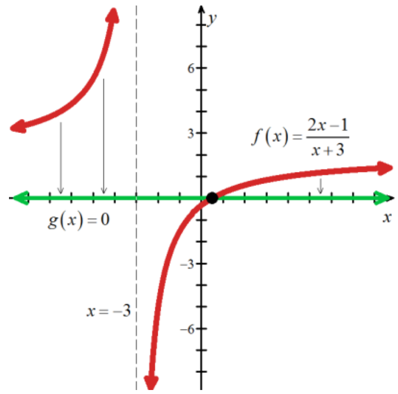
Solution. As in **Example 3.4.1**, we set $f(x)=\frac{2x-1}{x+3}$ and $g(x)=0$. The solutions to $f(x)=g(x)$ are the *x*-coordinates of the points where the graphs of $y=f(x)$ and $y=g(x)$ intersect. The solution to $f(x)\geq g(x)$ represents not only where the graphs meet, but the intervals over which $f(x)>g(x)$, or the graph of $y=f(x)$ is above the graph of $y=g(x)$. We show solutions graphically below.

Figure 3.4. 4



Graphs intersect at $\left(\frac{1}{2},0\right)$

Figure 3.4. 5

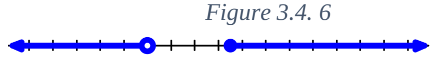


f is above g on $(-\infty,-3)$ and $\left(\frac{1}{2},\infty\right)$

We determined algebraically in **Example 3.4.1** that the two graphs intersect at the point where $x=\frac{1}{2}$.

Looking at the graphs of the two functions, we see that the graph of $y=f(x)$ is above the graph of $y=g(x)$ on $(-\infty,-3)$, as well as $\left(\frac{1}{2},\infty\right)$. Putting these results together, our solutions are *x*-values less than -3 and *x*-values greater than or equal to $\frac{1}{2}$. The solution set in set-builder notation is $\left\{x \mid x<-3 \text{ or } x\geq\frac{1}{2}\right\}$; in interval notation this is $(-\infty,-3)\cup\left[\frac{1}{2},\infty\right)$. Graphically, the solution set is shown below.

Figure 3.4. 6



□

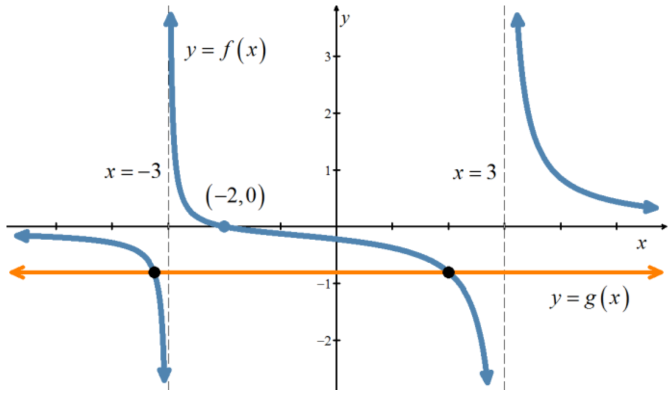
A few notes before moving on:

- While geometric interpretations of solutions to equations and inequalities are useful, in most cases some algebraic computation is required to verify these solutions.
- For the remainder of this section, we will use interval notation to express the solution.

Example 3.4.4. Solve $\frac{x+2}{x^2-9}\geq-\frac{4}{5}$ graphically.

Solution. Let $f(x)=\frac{x+2}{x^2-9}$ and $g(x)=-\frac{4}{5}$. We graph the function $f(x)=\frac{x+2}{x^2-9}$, with *x*-intercept $(-2,0)$ and vertical asymptotes $x=-3$ and $x=3$. The graph of $g(x)=-\frac{4}{5}$ is a horizontal line.

Figure 3.4. 7



$$y = f(x) = \frac{x+2}{x^2-9} \text{ and } y = g(x) = -\frac{4}{5}$$

To solve $f(x) \geq g(x)$, we find where the graphs of f and g intersect, $f(x) = g(x)$, and where the graph of f is above the graph of g, $f(x) > g(x)$. We start with the intersection.

$$\begin{aligned}
f(x) &= g(x) \\
\frac{x+2}{x^2-9} &= -\frac{4}{5} \\
5(x+2) &= -4(x^2-9), \text{ if } x^2-9 \neq 0 \\
5x+10 &= -4x^2+36 \\
4x^2+5x-26 &= 0 \\
(4x+13)(x-2) &= 0
\end{aligned}$$

After setting each factor equal to zero, we find potential solutions $x = -\frac{13}{4}$ and $x = 2$. Since neither makes the denominator, x^2-9, equal to zero, both $x = -\frac{13}{4}$ and $x = 2$ are solutions. Of course, we can also see that there are two solutions when we look at the graph.

We next check for solutions where the graph of f is above the graph of g. This includes *x*-values to the left of $-\frac{13}{4}$, *x*-values between the vertical asymptote $x = -3$ and 2, and *x*-values to the right of the vertical asymptote $x = 3$.

The solution set is $\left(-\infty, -\frac{13}{4}\right] \cup (-3, 2] \cup (3, \infty)$.

□

Steps for Solving a Rational Inequality Graphically

Consider the rational inequality $f(x) < g(x)$, $f(x) \leq g(x)$, $f(x) > g(x)$, or $f(x) \geq g(x)$.

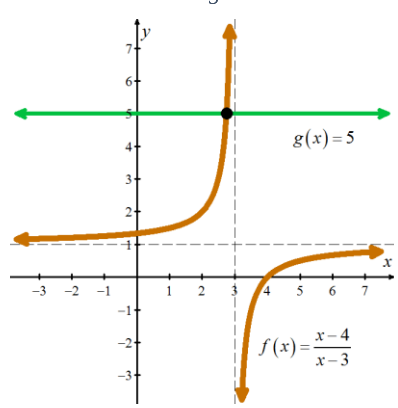
1. Graph $y = f(x)$ and $y = g(x)$.
2. Find the point(s) of intersection algebraically by solving $f(x) = g(x)$.
3. (a) The solution to $f(x) < g(x)$ is the set of x-values where the graph of $y = f(x)$ is below the graph of $y = g(x)$.
   (b) The solution to $f(x) \leq g(x)$ is the set of x-values where the graph of $y = f(x)$ intersects or is below the graph of $y = g(x)$.
   (c) The solution to $f(x) > g(x)$ is the set of x-values where the graph of $y = f(x)$ is above the graph of $y = g(x)$.
   (d) The solution to $f(x) \geq g(x)$ is the set of x-values where the graph of $y = f(x)$ intersects or is above the graph of $y = g(x)$.

Example 3.4.5. Solve $\dfrac{x-4}{x-3} < 5$ graphically.

Solution.

1. We graph $f(x) = \dfrac{x-4}{x-3}$ and $g(x) = 5$, following the usual steps for graphing a rational function and a line.

Figure 3.4. 8



2. To find the point(s) of intersection, we solve $f(x) = g(x)$, or $\dfrac{x-4}{x-3} = 5$.

$$\begin{aligned}\frac{x-4}{x-3}&=5\\ x-4&=5(x-3),\text{ if } x-3\neq 0\\ x-4&=5x-15\\ -4x&=-11\\ x&=\frac{11}{4}\end{aligned}$$

3. The solution of $f(x)<g(x)$ is the set of *x*-values where the graph of $y=f(x)$ is below the graph of $y=g(x)$. Since the graph of $y=f(x)$ is below the graph of $y=g(x)$ to the left of the point of intersection, $\left(\frac{11}{4},5\right)$, and to the right of the vertical asymptote, $x=3$, we find the solution set to be $\left(-\infty,\frac{11}{4}\right)\cup(3,\infty)$.

□

Just as we have relied on algebra to some extent in solving rational inequalities graphically, we will use our knowledge of graphs of rational functions in solving rational inequalities algebraically.

Solving Rational Inequalities Algebraically

We begin with the algebraic solution to the inequality presented in **Example 3.4.5**.

Example 3.4.6. Solve $\frac{x-4}{x-3}<5$ algebraically.

Solution. For a fully algebraic solution to the inequality $\frac{x-4}{x-3}<5$, we must avoid the temptation to multiply both sides by $(x-3)$, as we did when solving the equation. The problem is that, depending on x, $(x-3)$ may be positive or it may be negative. If $(x-3)$ is positive, multiplying by $(x-3)$ does not affect the inequality, but if $(x-3)$ is negative, multiplying by $(x-3)$ would reverse the inequality. Instead of working with these two separate cases, we collect all of the terms on the left side of the inequality with 0 on the right side, and then determine the sign of the left side.

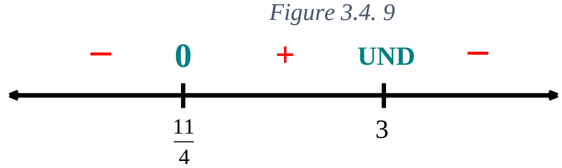
$$\frac{x-4}{x-3}<5$$
$$\frac{x-4}{x-3}-5<0$$
$$\frac{x-4-5(x-3)}{x-3}<0$$
$$\frac{-4x+11}{x-3}<0$$

We let $r(x)=\dfrac{-4x+11}{x-3}$ represent the left side of the inequality. The only value excluded from the domain of $r(x)$ is $x=3$. The value $x=\dfrac{11}{4}$, the solution of $-4x+11=0$, is in the domain and so it is the zero of this function. To determine the sign of r in intervals bounded by the zero, $x=\dfrac{11}{4}$, and the vertical asymptote, $x=3$, we select test values in each interval.

Interval	Test Value	Function Value	Sign
$\left(-\infty,\frac{11}{4}\right)$	$x=0$	$r(0)=-\frac{11}{3}$	$-$
$\left(\frac{11}{4},3\right)$	$x=\frac{23}{8}$	$r\left(\frac{23}{8}\right)=4$	$+$
$(3,\infty)$	$x=4$	$r(4)=-5$	$-$

We next construct a sign diagram in which we denote zeros with '0'. Vertical asymptotes and other *x*-values that are excluded from the domain have the symbol 'UND' for 'undefined'.

Figure 3.4. 9



We find $r(x)<0$, where we have the '−' sign, on the intervals $\left(-\infty,\frac{11}{4}\right)$ and $(3,\infty)$, so our solution set is $\left(-\infty,\frac{11}{4}\right)\cup(3,\infty)$.

□

Steps for Solving a Rational Inequality Algebraically

1. Rewrite the rational inequality as $r(x)<0$, $r(x)\le 0$, $r(x)>0$, or $r(x)\ge 0$, by moving all non-zero terms to the left side and simplifying.
2. Find the values at which $r(x)$ is undefined or zero. Equivalently, find the values excluded from the domain of r and the *x*-coordinates of the *x*-intercepts.
3. Place these values on a real number line. Write 'UND' or '0' above the values at which $r(x)$ is undefined or zero, respectively. This divides the real number line into subintervals in which $r(x)$ has the same sign. Test an *x*-value in each subinterval to find the sign of $r(x)$ and record it above each interval as '**−**' or '**+**'. This determines the sign of $r(x)$ everywhere.
4. Choose the *x*-values that correspond to the inequality in step 1 for the final solution.

We often shorten step 3 by just saying 'form a sign diagram for $r(x)$'.

Example 3.4.7. Solve $\dfrac{x+2}{x^2-9} \ge -\dfrac{4}{5}$ algebraically.

Solution.

1. We rewrite the inequality to get 0 on the right side.

$$\frac{x+2}{x^2-9} \ge -\frac{4}{5}$$
$$\frac{x+2}{x^2-9} + \frac{4}{5} \ge 0$$
$$\frac{5(x+2)+4(x^2-9)}{5(x^2-9)} \ge 0$$
$$\frac{4x^2+5x-26}{5(x^2-9)} \ge 0$$
$$\frac{(4x+13)(x-2)}{5(x-3)(x+3)} \ge 0$$

2. Letting $r(x)=\dfrac{(4x+13)(x-2)}{5(x-3)(x+3)}$, we note that $r(x)$ is undefined when $x=-3$ or $x=3$, and that $r(x)=0$ when $x=-\dfrac{13}{4}$ or when $x=2$.

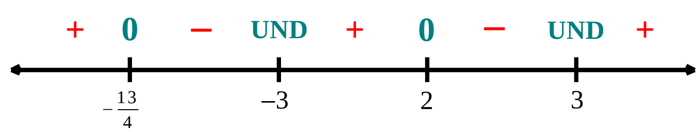
3. We begin by finding the sign in each interval determined by these values.

Interval	Test Value	Function Value	Sign
$\left(-\infty,-\frac{13}{4}\right)$	$x=-5$	$r(-5)=0.6125$	$+$
$\left(-\frac{13}{4},-3\right)$	$x=-\frac{25}{8}$	$r\left(-\frac{26}{8}\right)\approx-0.6694$	$-$
$(-3,2)$	$x=1$	$r(1)=0.425$	$+$
$(2,3)$	$x=\frac{5}{2}$	$r\left(\frac{5}{2}\right)\approx-0.8364$	$-$
$(3,\infty)$	$x=5$	$r(5)=1.2375$	$+$

We could achieve similar results by noting that each of the zeros in the numerator and denominator of $r(x)$ has multiplicity one. Thus, $r(x)$ will change sign across each of these values, so we could determine all of the signs by finding the sign in only one interval.

For the sign diagram, we place the *x*-values on the real number line, indicating the corresponding function values as zero or undefined, and add the appropriate signs above each interval.

Figure 3.4. 10



4. We are interested in where $r(x)\geq 0$. This occurs when $x\leq-\frac{13}{4}$, $-3<x\leq 2$, or $x>3$. Therefore, the solution set is $\left(-\infty,-\frac{13}{4}\right]\cup(-3,2]\cup(3,\infty)$.

□

Our last example has expressions of x on both sides of the inequality. Since the problem does not state a graphical method of solution, it will be solved algebraically.

Example 3.4.8. Solve $\dfrac{2x^2-2x-14}{x-2}\leq x+1$.

Solution.

1. We begin by collecting all non-zero terms on the left side so that we have 0 on the right side.

$$\frac{2x^2-2x-14}{x-2}-x-1\le 0$$

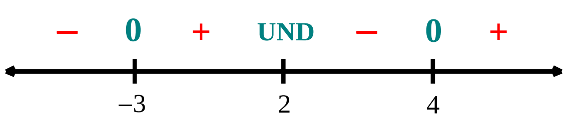
Next, we find a common denominator on the left side, and combine terms to get a single, simplified, rational expression.

$$\frac{2x^2-2x-14}{x-2}-x-1\le 0$$
$$\frac{2x^2-2x-14-x(x-2)-1(x-2)}{x-2}\le 0$$
$$\frac{x^2-x-12}{x-2}\le 0$$

2. Now, $r(x)=\dfrac{x^2-x-12}{x-2}=\dfrac{(x-4)(x+3)}{x-2}$. The domain excludes $x=2$, and the zeros are $x=-3$ and $x=4$.

3. We compose a sign chart by testing values in each interval.[14]

Figure 3.4. 11



We want to find where $r(x)\le 0$. On the intervals $(-\infty,-3)$ and $(2,4)$, we see that $r(x)<0$. After adding these intervals to the zeros −3 and 4, we get the solution set $(-\infty,-3]\cup(2,4]$.

□

[14] We have not included a table of test values here; try this on your own! As a time saver, note that we are only interested in signs, not precise values so estimating can save a lot of time. Additionally, taking advantage of multiplicities will lessen the number of intervals in which you need to calculate signs.

3.4 Exercises

1. Give an example of a rational equation that does not have a solution.

2. Give an example of a rational inequality that has a single solution.

In Exercises 3 – 8, solve the rational equation. Check domains to eliminate extraneous solutions.

3. $\dfrac{2x-3}{x+4}=0$

4. $\dfrac{2x-3}{x+4}=5$

5. $\dfrac{x}{5x+4}=3$

6. $\dfrac{3x-1}{x^2+1}=1$

7. $\dfrac{2x+17}{x+1}=x+5$

8. $\dfrac{x^2-2x+1}{x^3+x^2-2x}=1$

In Exercises 9 – 34, solve the rational inequality. Express your answer using interval notation.

9. $\dfrac{2}{x+1}>0$

10. $\dfrac{1}{x+2}\geq 0$

11. $\dfrac{4}{2x-3}\leq 0$

12. $\dfrac{2}{(x-1)(x+2)}<0$

13. $\dfrac{x-3}{x+2}\leq 0$

14. $\dfrac{x+2}{(x-1)(x-4)}\geq 0$

15. $\dfrac{(x+3)^2}{(x-1)^2(x+1)}>0$

16. $\dfrac{x}{x^2-1}>0$

17. $\dfrac{4x}{x^2+4}\geq 0$

18. $\dfrac{x^2-x-12}{x^2+x-6}>0$

19. $\dfrac{3x^2-5x-2}{x^2-9}<0$

20. $\dfrac{x^3+2x^2+x}{x^2-x-2}\geq 0$

21. $\dfrac{x^2+5x+6}{x^2-1}>0$

22. $\dfrac{3x-1}{x^2+1}\leq 1$

23. $\dfrac{2x-1}{x+3}\leq -5$

24. $\dfrac{3x+5}{x-4}<2$

25. $\dfrac{x+5}{x-3}\leq -4$

26. $\dfrac{3x+1}{x-2}>5$

27. $\dfrac{2x+17}{x+1} > x+5$

28. $\dfrac{1}{x^2+1} < 0$

29. $\dfrac{x^4-4x^3+x^2-2x-15}{x^3-4x^2} \ge x$

30. $\dfrac{5x^3-12x^2+9x+10}{x^2-1} \ge 3x-1$

31. $\dfrac{6}{x^2-x-2} \ge \dfrac{x+3}{x^2-1}$

32. $\dfrac{1}{x-3}+\dfrac{1}{x+3} < \dfrac{10}{x^2-9}$

33. $\dfrac{1}{x^2-x-6}+\dfrac{2}{x^2+2x} \le \dfrac{15}{x^2-3x}$

34. $\dfrac{6}{x+1}+\dfrac{x^2-5x}{x^3+x^2-x-1} \ge 1+\dfrac{5}{x^2+2x+1}$

35. Given $f(x)=\sqrt{2-4x}$ and $g(x)=-\dfrac{3}{x}$, find the domain of the composite function $(f\circ g)(x)$.

36. The population of Sasquatch in Salt Lake County was modeled by the function $P(t)=\dfrac{150t}{t+15}$, where $t=0$ represents the year 1803. According to this model, when were there fewer than 100 Sasquatch in Salt Lake County?

37. Following surgery, a patient has been receiving a pain relief medication intravaniously. The concentration C (in milligrams per liter) of the medication in the patient's bloodstream t hours after this process started is given by $C(t)=\dfrac{50t+10}{t+2}$, $t \ge 0$.

a) The patient will not receive pain relief unless the concentration of the medication is 20 or more milligrams per liter. Use this function to determine the time interval for which the concentration of the medication, C, will be greater than or equal to 20 milligrams per liter.

b) As time goes on (as $t \to \infty$), what value will the concentration of the medication approach?

38. At a manufacturing plant that assembles clock radios, each of the assembly workers is responsible for assembling an entire radio from start to finish. Due to turnover in the workforce, new assembly workers are being hired on a regular basis. Based on past performance, the manager of the plant has determined that the number of radios, N, assembled by a worker each week after t weeks of working at the plant is given by $N(t)=\dfrac{45t+6}{3t+2}$, $t \ge 0$.

a) When an employee is able to assemble $N=12$ radios in a week, their probation period is ended and they are given a pay raise. According to this model, at what time t will the new employee be able to assemble 12 radios in a week?

b) According to this model, what is the limiting value for the number of radios an employee can assemble in a week as time increases, that is, as $t \to \infty$?

CHAPTER 4
EXPONENTIAL AND LOGARITHMIC FUNCTIONS

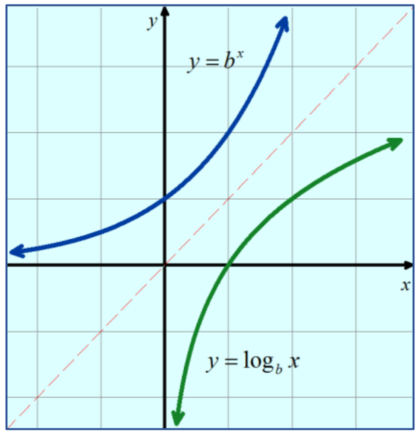


Figure 4.0. 1

Chapter Outline

Introduction

In this chapter, we investigate exponential and logarithmic functions, equations, graphs, and applications. The first two sections in the chapter review ideas explored in Intermediate Algebra and lay the foundation for the last three sections where you should solidify your skills in solving, graphing, and applying these functions and equations. Other key ideas in this chapter are a) the nature of the relationship between exponential and logarithmic functions (they are inverses of one another), b) how ideas about graphing, properties, and solving equations are interrelated, and c) the prevalence of exponential and logarithmic relationships in the world around us.

In Section 4.1 you are asked to think about the output of an exponential functions at 'non-standard' values, for example evaluating $f(x) = 2^x$ for $x = 3.1$ or $x = \pi$. The inquiry leads to the understanding that for basic exponential functions (in the form of $f(x) = b^x$ and $b > 0$, $b \neq 1$), the domain is all real numbers, the range is $(0,\infty)$, exponential functions are one-to-one, and the *y*-intercept is $(0,1)$. From this, two major ideas are developed a) basic rules of graphing transformations apply to exponential functions, and b) the fact that exponential functions are one-to-one allows us to solve many basic equations involving exponents where a common base can be found. The latter then leads to developing an understanding of the equivalence relationship between logarithms and exponentials, how the graphs of the two basic functions ($f(x)=b^x$ and $f(x)=\log_b(x)$) are reflections of one another across the line $y=x$, and finally that the domain of a basic logarithmic function is $(0,\infty)$, while the range is all real numbers (because logarithmic and exponential functions are inverses of each other.)

In Section 4.2, you continue to explore logarithmic functions and equations. The beginning of the section is devoted to building an understanding of logarithmic properties around changes of base. The section then moves to using properties of logarithms to solve equations involving exponents where a common base cannot be found. At the end of the section, you explore how to simplify and expand logarithmic expressions and how this skill applies to solving logarithmic equations.

Section 4.3 solidifies ideas introduced in 4.1 and 4.2 about exponential equations and functions. By the end of this section, you should be able to solve a wide variety of exponential equations. You should also be able to graph a variety of exponential functions involving transformations, and be able to state domain, range (a great deal of emphasis is placed here), and *x*- and *y*-intercepts of the transformed functions.

Section 4.4 also solidifies ideas from 4.1 and 4.2, but focuses on logarithmic functions. By the end of this section, you should be able to solve a wide variety of logarithmic equations. You should also be able to graph logarithmic functions involving an assortment of transformations, state the *x*- and *y*-intercept of the graphs as well as the domain (a great deal of emphasis is placed here), and identify the range.

Section 4.5 focuses on applications of logarithmic and exponential functions such as compounding interest, uninhibited growth and/or decay, and cooling/heating problems. Attention should be paid in this section to using ideas developed in earlier sections. For example, you should understand when using the Exponential Growth or Decay formula $A(t)=A_0e^{kt}$ that if k is negative, $A(t)$ will be decreasing, or that if t is 0, then $A(0)=A_0$, the starting condition. You should also have a solid idea of how to use your understanding of properties of logarithmic or exponential functions to manipulate or develop formulas for solving problems.

4.1 Introduction to Exponential and Logarithmic Functions

Learning Objectives

- Evaluate exponential expressions and functions.
- Graph basic exponential functions, including transformations.
- Use the one-to-one property to solve common-base exponential equations.
- Evaluate logarithmic expressions and functions.
- Solve logarithmic equations by conversion to exponential form.
- Graph basic logarithmic functions, including transformations.

Up to this point, we have dealt with functions that involve terms like x^2 or $x^{\frac{2}{3}}$; in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x)=b^x$ where the base b is a constant and the exponent x is the variable. This first section introduces us to exponential functions and logarithmic functions while the rest of the chapter will explore their properties and applications. We begin with a quick review of exponents, and revisit some basic properties.

Example	General Definition
$2^3=2\cdot 2\cdot 2=8$	$a^n=\underbrace{a\cdot a\cdot a\cdots a}_{n\text{ times}}$, $n=1, 2, 3, \ldots$
$16^{\frac{1}{4}}=2$ since $2^4=16$ and $2>0$	If $n=2, 4, 6, \ldots$ and $a\geq 0$, then $a^{\frac{1}{n}}=\sqrt[n]{a}=b$, where $b^n=a$ and $b\geq 0$.
$(-8)^{\frac{1}{3}}=-2$ since $(-2)^3=-8$	If $n=1, 3, 5, \ldots$ then $a^{\frac{1}{n}}=\sqrt[n]{a}=b$ where $b^n=a$.
$8^{\frac{2}{3}}=\left(8^{\frac{1}{3}}\right)^2=2^2=4$	For $\dfrac{m}{n}$ in lowest terms, $n\neq 0$, $a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^m=(a^m)^{\frac{1}{n}}$ if $a^{\frac{1}{n}}$ is a real number.

If we cannot find the exact value of an exponential term, we can estimate it. For example, $85^{\frac{1}{4}}$ is just a bit more than 3 since $3^4=81$. Using a calculator we get $85^{\frac{1}{4}}\approx 3.03637$. What if the exponent is not a rational number, for example 2^{π}? We will not formally define irrational exponents. However, since

$\pi \approx 3.14159$ and, as we will learn shortly, $y = 2^x$ is a continuously increasing function, we know that $2^3 < 2^\pi < 2^4$, or equivalently $8 < 2^\pi < 16$. In fact, by using a calculator to evaluate, we find that $2^\pi \approx 8.82498$.

Knowing that it is possible to evaluate exponential terms with either rational or irrational exponents, we move on to discussing exponential functions and their applications to our daily lives.

Example 4.1.1. The value of a car can be modeled by $V(x) = 25(0.8)^x$, where $x \geq 0$ is the age of the car in years and $V(x)$ is the value in thousands of dollars. Find and interpret $V(0)$ and $V(7)$.

Solution. To find $V(0)$, we replace x with 0 to obtain $V(0) = 25(0.8)^0 = 25$. To find $V(7)$, we replace x with 7 and have $V(7) = 25(0.8)^7 = 5.24288$. Since x represents the age of the car in years and $V(x)$ is measured in thousands of dollars, $V(0) = 25$ tells us that the purchase price of the car was \$25,000, while $V(7) = 5.24288$ tells us that the value of the car after seven years is about \$5,243.

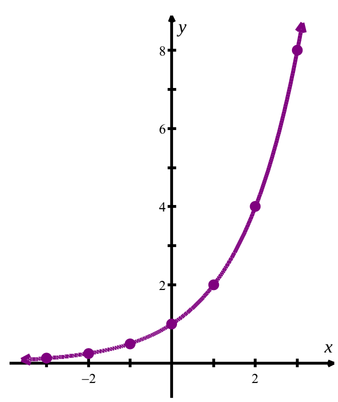
□

Basic Exponential Functions

We start our exploration of exponential functions with a graph of $f(x) = 2^x$. After creating a table of values, we plot and connect the points with a smooth curve.

x	$f(x) = 2^x$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0,1)$
1	$2^1 = 2$	$(1,2)$
2	$2^2 = 4$	$(2,4)$
3	$2^3 = 8$	$(3,8)$

Figure 4.1. 1



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ are in order.

- The domain of f is $(-\infty, \infty)$, the range is $(0, \infty)$, and the point $(0,1)$ is the *y*-intercept.

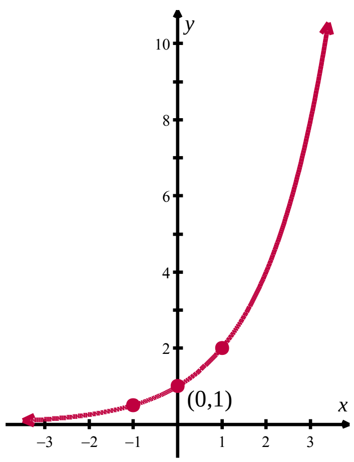
- The line $y=0$, the x-axis, is a horizontal asymptote since, as $x\to-\infty$, $f(x)\to 0$. Note that the end behavior of this function is different from rational functions in that the graph approaches the x-axis to the left, but not to the right.
- As x becomes larger, $f(x)$ grows larger as well, in fact very quickly. This is called exponential growth, due to the fact that x appears in the exponent.
- The graph of f passes the horizontal line test, which means f is one-to-one. That is, by **Definition 1.13**, if $2^m=2^n$ then $m=n$. We will use this property in solving exponential equations. Another consequence of $f(x)=2^x$ being one-to-one is that it is invertible (see **Theorem 1.3**). Later in this section we will see that the inverse function is called a logarithmic function.

We proceed with a definition of the function, $f(x)=b^x$. We do not include negative values for b since they may result in non-real values for b^x, such as $(-4)^{1/2}=\sqrt{-4}=2i$. We exclude $b=0$ since 0^0 is undefined, and we exclude $b=1$ since $f(x)=1^x=1$ is equivalent to the constant function $y=1$.

Definition 4.1. A function of the form $f(x)=b^x$ where b is a positive real number, $b\neq 1$, is called an **exponential function with base b**.

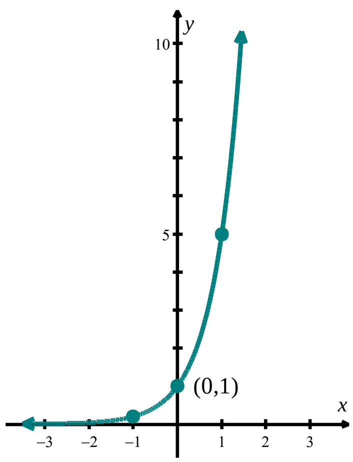
To get a better feel for the shape of exponential functions, we compare the following graphs.

Figure 4.1. 2



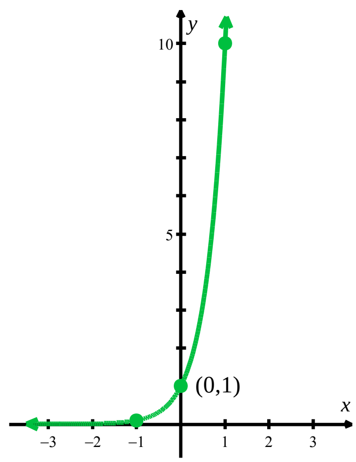
$y=f(x)=2^x$

Figure 4.1. 3



$y=g(x)=5^x$

Figure 4.1. 4

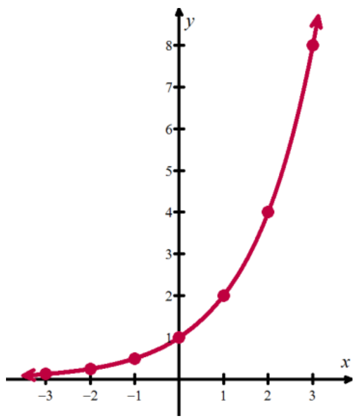


$y=h(x)=10^x$

We see that all three graphs share the same domain and range, basic shape, horizontal asymptote, and y-intercept of $(0,1)$. We also see that the larger the base value, the faster the graph approaches its horizontal asymptote on the left side and the steeper/faster the growth is on the right side.

What if $0 < b < 1$? Consider $F(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $F(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$ where $f(x) = 2^x$. Thinking back to **Section 1.3**, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the *y*-axis.[1]

Figure 4.1. 5

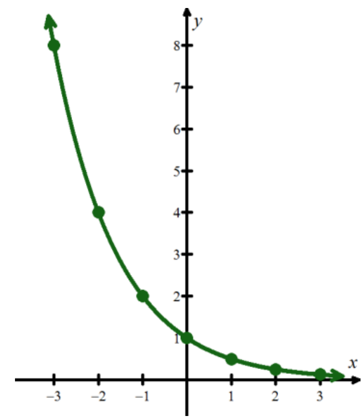


$y = f(x) = 2^x$

reflect across *y*-axis

multiply each *x*-coordinate by −1

Figure 4.1. 6



$y = F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

We see that the domain and range of F match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, F is one-to-one, but while f is always increasing, F is always decreasing. We summarize the basic properties of exponential functions, as follows.

[1] Try creating a table of values for $F(x) = \left(\frac{1}{2}\right)^x$ and $G(x) = 2^{-x}$ to verify this relationship.

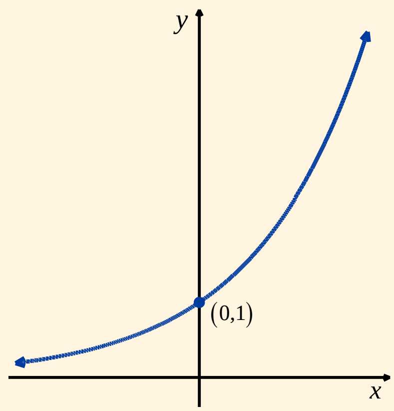
Properties of the Exponential Function $f(x)=b^x$, with $b>0$ and $b\neq 1$

- The domain of f is $(-\infty,\infty)$ and the range is $(0,\infty)$.
- The graph of f has a y-intercept at the point $(0,1)$, and its horizontal asymptote is $y=0$.
- f is a one-to-one function. **The one-to-one property of exponential functions**: if $b^m=b^n$ then $m=n$.

If $b>1$,

- f is always positive and increasing.
- As $x\to-\infty$, $f(x)\to 0$.
- As $x\to\infty$, $f(x)\to\infty$.

Figure 4.1. 7

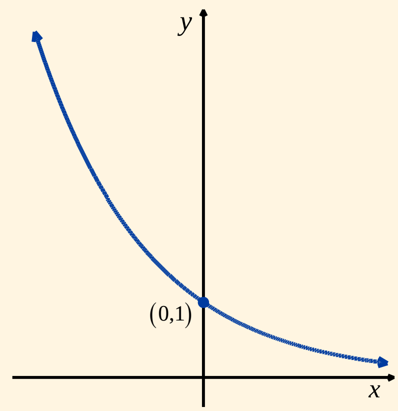


$y=b^x$, $b>1$

If $0<b<1$,

- f is always positive and decreasing.
- As $x\to-\infty$, $f(x)\to\infty$.
- As $x\to\infty$, $f(x)\to 0$.

Figure 4.1. 8



$y=b^x$, $0<b<1$

Graphing Basic Exponential Functions Using Transformations

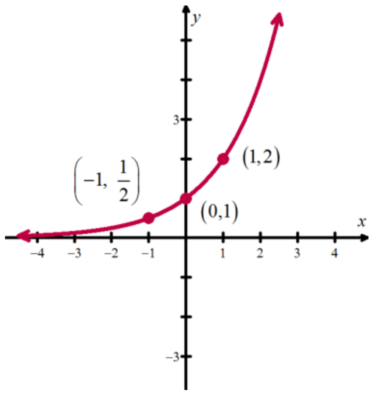
If we think of the **basic exponential function** $y=b^x$ as a new toolkit, or parent, function, we can use results from **Section 1.3** to graph transformations of $y=b^x$, as in the next example.

Example 4.1.2. Graph $f(x)=2^{x+1}-3$ by using transformations. State the domain, range, and asymptote.

Solution. We graph f through transformations of the function $y=2^x$. We start with a sketch of $y=2^x$ that passes through the points $\left(-1,\frac{1}{2}\right)$, $(0,1)$, and $(1,2)$, and approaches its horizontal asymptote of $y=0$. We then apply a sequence of transformations that result in the graph of $y=f(x)=2^{x+1}-3$. Since the input changes from x to $x+1$, we subtract 1 from each of the x coordinates on the graph of

$y=2^x$, which shifts the graph to the left by one unit. The '-3' affects the output, and so we next subtract 3 from each *y*-coordinate, resulting in a shift down by 3 units.

Figure 4.1. 9

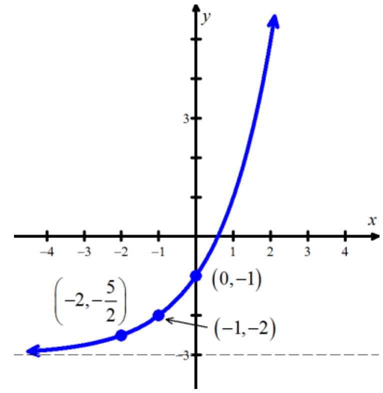


$y=2^x$

1. Subtract 1 from each *x*-coordinate;
shift left 1 unit.

2. Subtract 3 from each *y*-coordinate;
shift down 3 units

Figure 4.1. 10



$y=f(x)=2^{x+1}-3$

Since the domain of $y=2^x$ is the set of all real numbers, subtracting 1 from the *x*-coordinates does not change the domain. However, both the range, $(0,\infty)$, and the horizontal asymptote, $y=0$, are changed by shifting the graph down by 3 units. Thus, the domain of $f(x)=2^{x+1}-3$ is $(-\infty,\infty)$, its range is $(-3,\infty)$, and its horizontal asymptote is $y=-3$.

□

In the previous example, note that we used the points $\left(-1,\dfrac{1}{2}\right)$, $(0,1)$, and $(1,2)$ to graph $y=2^x$. It is a good practice to use the three *x*-values -1, 0, and 1 in graphing basic exponential functions.

Using the One-to-One Property to Solve Exponential Equations

Suppose, for instance, we want to solve the equation $2^x=128$. Since $128=2^7$, we can write $2^x=128$ as $2^x=2^7$. The one-to-one property of exponential functions, ' if $b^m=b^n$ then $m=n$ ', tells us that the solution is $x=7$.

Example 4.1.3. Solve the following equations using the one-to-one property of exponential functions.

1. $9^x=\dfrac{1}{27}$

2. $2^{3x}=16^{1-x}$

Solution.

1. To apply the one-to-one property to $9^x = \frac{1}{27}$, we look for a common base. Noting that both 9 and 27 are powers of 3, we write each side of the equation with base 3.

$$\begin{aligned} 9^x &= \frac{1}{27} \\ \left(3^2\right)^x &= \frac{1}{3^3} \\ 3^{2x} &= 3^{-3} \end{aligned}$$

From the one-to-one property of exponential functions, we find $2x = -3$, from which $x = -\frac{3}{2}$.

2. Since 16 is a power of 2, we can rewrite the equation $2^{3x} = 16^{1-x}$ as follows.

$$\begin{aligned} 2^{3x} &= \left(2^4\right)^{1-x} \\ 2^{3x} &= 2^{(4)(1-x)} \end{aligned}$$

Using the one-to-one property of exponential functions, $3x = 4(1-x)$, from which $x = \frac{4}{7}$.

□

Definition of Logarithms

We begin with the observation that $2^x = 8$ means x is the exponent on 2 that results in 8, or 'the power of 2 that gives 8'. We adopt a special notation for this x-value: $\log_2 8$, read as 'logarithm with base 2 of 8'. Since we can see that $2^3 = 8$, we conclude that $\log_2 8 = 3$.

Definition 4.2. For $y > 0$ and b a positive number other than 1, $\log_b y$, called a **logarithm with base b of y**, is the power of b that gives y.

Note that it is common to say 'log' in place of 'logarithm'.

Example 4.1.4. Find the exact values of the following logarithms.

1. $\log_3 81$
2. $\log_5\left(\frac{1}{25}\right)$
3. $\log_8 4$

Solution.

1. Since $\log_3 81$ is the power of 3 that gives 81, we check powers of 3: $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$. We see that the answer is 4. That is, $\log_3 81 = 4$ since $3^4 = 81$.

2. For $\log_5\left(\frac{1}{25}\right)$, we look for the power of 5 that gives $\frac{1}{25}$. Since $\frac{1}{25}=\frac{1}{5^2}=5^{-2}$, we find

   $\log_5\left(\frac{1}{25}\right)=-2$.

3. To determine the value of $\log_8 4$, we search for the power of 8 that gives 4. This is a bit harder to guess, so we look for the value of x for which $\log_8 4=x$, or equivalently $8^x=4$. We can then solve this equation using the one-to-one property of exponential functions.

$$\begin{aligned} 8^x &= 4 \\ \left(2^3\right)^x &= 2^2 \\ 2^{3x} &= 2^2 \end{aligned}$$

   We find $3x=2$, from which $x=\frac{2}{3}$. So, $\log_8(4)=\frac{2}{3}$. We saw in the first part of this section that $8^{\frac{2}{3}}=\left(8^{\frac{1}{3}}\right)^2=2^2=4$, which verifies this answer.

□

While we cannot find the exact value of every logarithm, as in the previous example, we can estimate values of logarithms. For example, $\log_3 85$ is just a bit more that 4, since $3^4=81$ and 85 is a bit more than 81. The only logarithms that we can find the exact value of are those of the form $\log_b y$ for which the argument y is a power of the base b.

We note that the equations $\log_3 81=4$ and $3^4=81$ contain the same information. The first is stated in 'logarithmic form' and the second in 'exponential form'. This equivalency is stated, in general, as follows.

Equivalency of Logarithmic and Exponential Equations

For $y>0$ and b a positive constant other than 1, $\log_b y=x \Leftrightarrow b^x=y$.

The equation $\log_b y=x$ is equivalent to the equation $b^x=y$. We say $\log_b y=x$ is the **logarithmic form** and $b^x=y$ is the **exponential form**. Switching between logarithmic form and exponential form is useful in solving logarithmic and exponential equations. Keep in mind the mnemonic: 'a log is an exponent'. The following is helpful in identifying the locations of base and exponent in each of these forms.

$$\overset{\text{exponent}}{\underset{\downarrow}{x}} = \log_{\underset{\text{base}}{\uparrow}{b}} y \quad \Leftrightarrow \quad y = \underset{\underset{\text{base}}{\uparrow}}{b}^{\overset{\text{exponent}}{\downarrow}{x}}$$

Example 4.1.5. Convert the following from logarithmic form to exponential form, or vice versa.

1. $\log_6 \sqrt{6} = \frac{1}{2}$
2. $10^{-3} = \frac{1}{1000}$

Solution.

1. Converting from logarithmic form to exponential form, $\log_6 \sqrt{6} = \frac{1}{2} \Rightarrow 6^{\frac{1}{2}} = \sqrt{6}$.
2. Converting from exponential form to logarithmic form, $10^{-3} = \frac{1}{1000} \Rightarrow \log_{10}\left(\frac{1}{1000}\right) = -3$.

□

Solving Logarithmic Equations by Conversion to Exponential Form

Example 4.1.6. Solve the following equations by first changing forms.

1. $\log_3 x = 4$
2. $\log_2 (x-1) = 3$

Solution.

1. We convert $\log_3 x = 4$ to exponential form to get $x = 3^4$, from which $x = 81$.
2. We start by converting $\log_2 (x-1) = 3$ to the exponential form $x - 1 = 2^3$. Then $x = 2^3 + 1$, for a final answer of $x = 9$.

□

Basic Logarithmic Functions

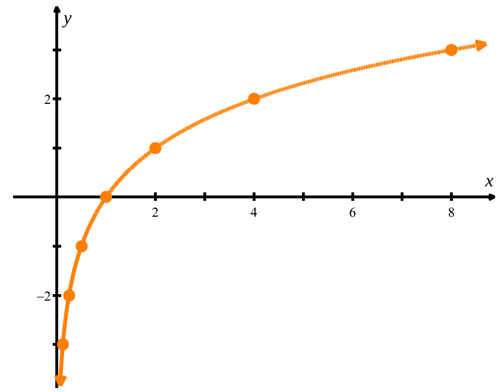
We are now ready for the definition of logarithmic functions.

> **Definition 4.3.** A function of the form $f(x) = \log_b x$ where b is a positive number, $b \neq 1$, and $x > 0$, is called a **logarithmic function with base *b***.

We begin our exploration of logarithmic functions with a graph of $g(x) = \log_2 x$. After creating a table of values, we plot and connect the points with a smooth curve. Since we can easily evaluate the logarithm for powers of the base 2, we choose such numbers for the *x*-values.

x	$g(x)=\log_2 x$	$(x,g(x))$
$2^{-3}=\frac{1}{8}$	$\log_2 2^{-3}=-3$	$\left(\frac{1}{8},-3\right)$
$2^{-2}=\frac{1}{4}$	$\log_2 2^{-2}=-2$	$\left(\frac{1}{4},-2\right)$
$2^{-1}=\frac{1}{2}$	$\log_2 2^{-1}=-1$	$\left(\frac{1}{2},-1\right)$
$2^{0}=1$	$\log_2 2^{0}=0$	$(1,0)$
$2^{1}=2$	$\log_2 2^{1}=1$	$(2,1)$
$2^{2}=4$	$\log_2 2^{2}=2$	$(4,2)$
$2^{3}=8$	$\log_2 2^{3}=3$	$(8,3)$

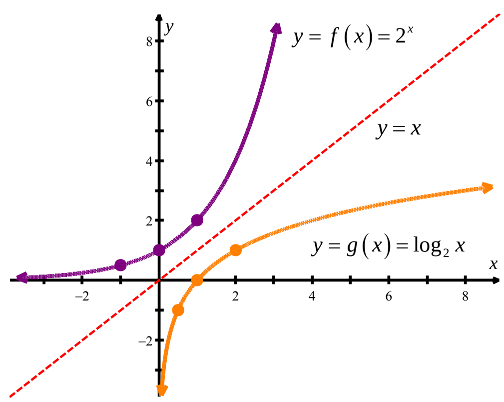
Figure 4.1. 11



$y=g(x)=\log_2 x$

Comparing the graph of $g(x)=\log_2 x$ with the graph of $f(x)=2^x$, shown at the beginning of this section, we see that the two graphs are symmetric across the line $y=x$, as indicated by the fact that the *x*- and *y*-coordinates of their points are interchanged. It follows that f and g are inverses of each other.[2]

Figure 4.1. 12



To show this more formally, we consider $(f\circ g)(x)$ and $(g\circ f)(x)$:

[2] Refer to **Section 1.5**.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\log_2 x) \\ &= 2^{\log_2 x} \\ &= x\end{aligned}$$

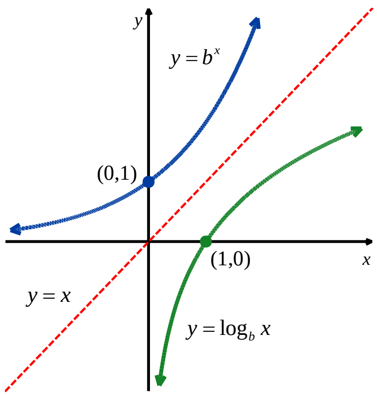
since $\log_2 x$ is the power of 2 that gives x

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2^x) \\ &= \log_2 2^x \\ &= x\end{aligned}$$

since x is the power of 2 that gives 2^x

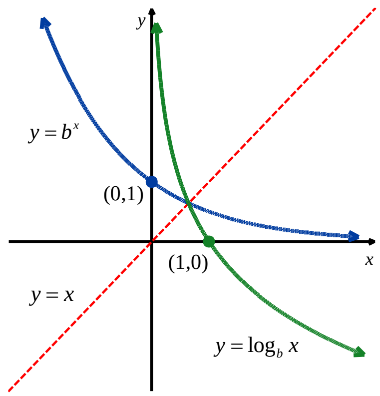
Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, we have shown f and g are inverses of each other.[3] In general, $f(x) = b^x$ and $g(x) = \log_b x$, $b > 0$, $b \neq 1$, are inverses of each other.[4] We use this inverse property to graph basic logarithmic functions by reflecting graphs of basic exponential functions across the line $y = x$.

Figure 4.1. 13



$b > 1$

Figure 4.1. 14



$0 < b < 1$

As seen from the graphs, the *x*-intercept of the logarithmic function $f(x) = \log_b x$ is the point $(1,0)$ and its vertical asymptote is the *y*-axis. These and other properties are summarized, as follows.

[3] See **Definition 1.12**.
[4] This general case is verified in **Section 4.2**.

Properties of the Logarithmic Function $f(x)=\log_b x$, with $b>0$ and $b\neq 1$

- The domain of f is $(0,\infty)$ and the range is $(-\infty,\infty)$.
- The graph of f has an x-intercept at the point $(1,0)$ and its vertical asymptote is $x=0$.
- f is a one-to-one function. **The one-to-one property of logarithmic functions**: if $\log_b u=\log_b v$ then $u=v$.

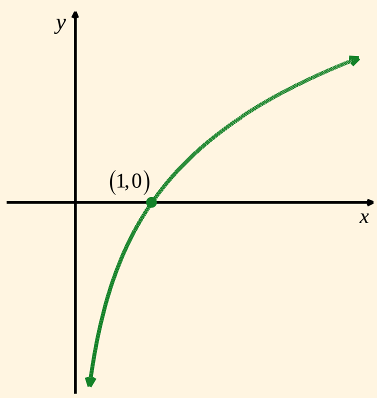
If $b>1$,

- f is always increasing.
- As $x\to 0^+$, $f(x)\to-\infty$.
- As $x\to\infty$, $f(x)\to\infty$.

If $0<b<1$,

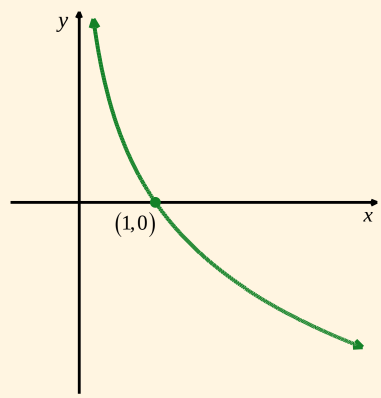
- f is always decreasing.
- As $x\to 0^+$, $f(x)\to\infty$.
- As $x\to\infty$, $f(x)\to-\infty$.

Figure 4.1. 15



$y=\log_b x$, $b>1$

Figure 4.1. 16



$y=\log_b x$, $0<b<1$

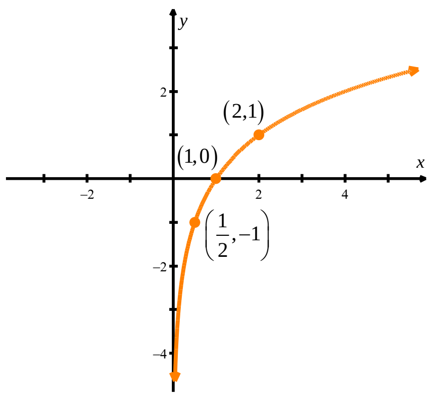
Graphing Basic Logarithmic Functions Using Transformations

If we think of the logarithmic function $f(x)=\log_b x$ as a new toolkit, or parent, function, we can use results from **Section 1.3** to graph transformations of $f(x)=\log_b x$, as in the next two examples.

Example 4.1.7. Graph $f(x)=\log_2(x+3)-1$. State the domain, range, and asymptote.

Solution. We graph f through transformations of the function $y=\log_2 x$. The change of input from x to $x+3$ means that we need to subtract 3 from each x-coordinate of points on the graph of $y=\log_2 x$; this shifts the graph to the left by three units. Subtracting 1 from the output means that we subtract 1 from each y-coordinate, or shift the graph down by one unit.

Figure 4.1. 17

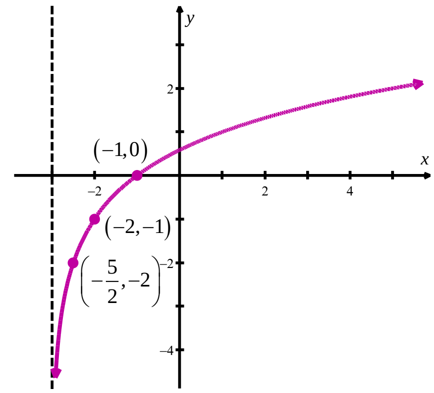


$y=\log_2 x$

1. Shift left 3 units.

2. Shift down 1 unit.

Figure 4.1. 18



$y=f(x)=\log_2(x+3)-1$

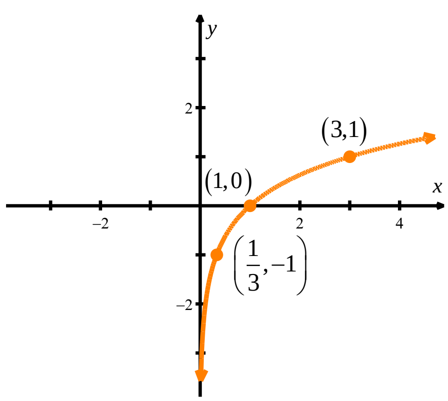
The domain of f is $(-3,\infty)$, its range is $(-\infty,\infty)$, and the vertical asymptote is $x=-3$.

□

Example 4.1.8. Graph $f(x)=1+\log_3(-x)$. State the domain, range, and asymptote.

Solution. We graph f through transformations of the function $y=\log_3 x$. The change of input from x to $-x$ means that we need to multiply each *x*-coordinate of the points on the graph of $y=\log_3 x$ by -1; this reflects the graph across the *y*-axis. Adding 1 to the output means that we next need to add one unit to each *y*-coordinate; this shifts the graph up by one unit.

Figure 4.1. 19

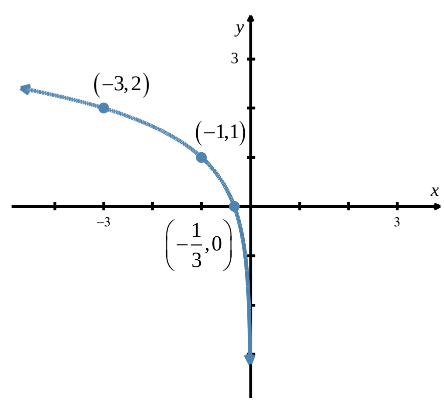


$y=\log_3 x$

1. Reflect across *y*-axis.

2. Shift up one unit.

Figure 4.1. 20



$y=f(x)=1+\log_3(-x)$

The domain of f is $(-\infty,0)$, its range is $(-\infty,\infty)$, and its vertical asymptote is the line $x=0$.

□

4.1 Exercises

1. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

2. Does the graph of a logarithmic function have a horizontal asymptote? Explain.

In Exercises 3 – 8, rewrite each equation in exponential form.

3. $\log_5 25 = 2$
4. $\log_{25} 5 = \frac{1}{2}$
5. $\log_3\left(\frac{1}{81}\right) = -4$
6. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$
7. $\log_b m = c$
8. $\log_5 a = 2$

In Exercises 9 – 14, rewrite each equation in logarithmic form.

9. $2^3 = 8$
10. $5^{-3} = \frac{1}{125}$
11. $4^{\frac{5}{2}} = 32$
12. $\left(\frac{1}{3}\right)^{-2} = 9$
13. $a^b = m$
14. $b^3 = c$

In Exercises 15 – 35, simplify the expression without using a calculator.

15. $\log_3 27$
16. $\log_6 216$
17. $\log_2 32$
18. $\log_a a^6$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_{27} 9$
21. $\log_{36} 216$
22. $\log_{\frac{1}{5}} 625$
23. $\log_{\frac{1}{6}} 216$
24. $\log_{\frac{1}{c}} c^2$
25. $\log_{36} 36$
26. $\log_4 8$
27. $\log_6 1$
28. $\log_c 1$
29. $\log_{13}\sqrt{13}$
30. $\log_{36}\sqrt[4]{36}$
31. $7^{\log_7 3}$
32. $36^{\log_{36} 216}$
33. $\log_{36} 36^{216}$
34. $\log_5 3^{\log_3 5}$
35. $\log_2 3^{\log_3 2}$

In Exercises 36 – 38, determine two integers that bound the value of the logarithmic expression; one integer that is smaller and one integer that is larger than the expression.

36. $\log_3 59$
37. $\log_4\left(\frac{1}{14}\right)$
38. $\log_{10} 900$

In Exercises 39 – 47, solve the equation using the one-to-one property of exponential functions.

39. $2^{4x} = 8$

40. $3^{x-1} = 27$

41. $5^{2x-1} = 125$

42. $4^{2x} = \frac{1}{2}$

43. $8^{x} = \frac{1}{128}$

44. $2^{x^3-x} = 1$

45. $3^{7x} = 81^{4-2x}$

46. $3^{7x+2} = \left(\frac{1}{9}\right)^{2x}$

47. $\left(\frac{1}{2}\right)^{3x} = 2^{x+4}$

In Exercises 48 – 56, solve the equation by converting the logarithmic equation to exponential form.

48. $\log_3 x = 2$

49. $\log_2 x = 6$

50. $\log_9 x = \frac{1}{2}$

51. $\log_6 x = -3$

52. $\log_2 x = -3$

53. $\log_3 x = 3$

54. $\log_{18} x = 2$

55. $\log_3 (7-2x) = 2$

56. $\log_{\frac{1}{2}} (2x-1) = -3$

In Exercises 57 – 62, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain, range, and asymptote of g.

57. $f(x) = 2^x$; $g(x) = 2^x - 1$

58. $f(x) = 2^x$; $g(x) = 2^{-x}$

59. $f(x) = 2^x$; $g(x) = -2^x + 3$

60. $f(x) = 2^x$; $g(x) = 2^{x-2}$

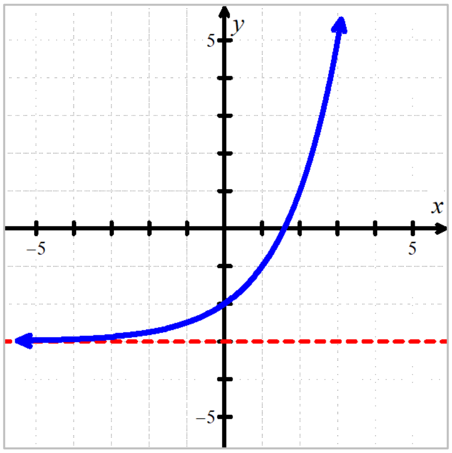
61. $f(x) = \left(\frac{1}{3}\right)^x$; $g(x) = \left(\frac{1}{3}\right)^{x-1}$

62. $f(x) = 3^x$; $g(x) = 3^{-x} + 2$

In Exercises 63 – 66, identify a function whose graph is shown. Asymptotes are drawn as dashed lines.

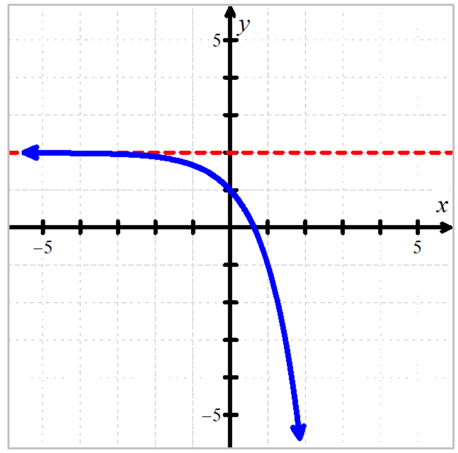
63.

Figure Ex4.1. 1



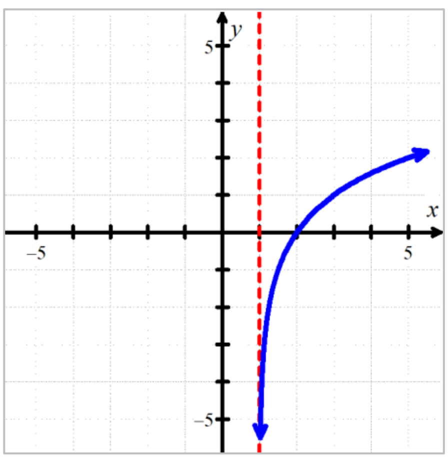
64.

Figure Ex4.1. 2



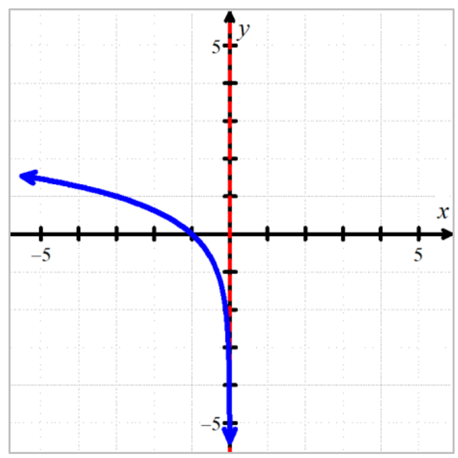
65.

Figure Ex4.1. 3



66.

Figure Ex4.1. 4



In Exercises 67 – 72, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain, range, and asymptote of g.

67. $f(x) = \log_2 x$; $g(x) = \log_2(x+1)$

68. $f(x) = \log_2 x$; $g(x) = \log_2(x-2)$

69. $f(x) = \log_{\frac{1}{3}} x$; $g(x) = \log_{\frac{1}{3}} x + 1$

70. $f(x) = \log_3 x$; $g(x) = -\log_3(x-2)$

71. $f(x) = \log_4 x$; $g(x) = 2\log_4 x$

72. $f(x) = \log_3 x$; $g(x) = 2\log_3(x+4) - 1$

(Logarithmic Scales) In Exercises 73 – 75, we introduce three widely used measurement scales that involve logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

73. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology or the U.S. Geological Survey's Earthquake Hazards Program. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a 'magnitude 0 event', which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log_{10}\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.[5]

(a) Show that $M(0.001) = 0$.

(b) Compute $M(80{,}000)$.

(c) Show that an earthquake that registered 6.7 on the Richter scale had a seismograph reading ten times larger than one that measured 5.7.

74. While the decibel scale can be used in many disciplines, we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound. The Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log_{10}\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to a baseline: $10^{-12}\dfrac{W}{m^2}$ is the threshold of human hearing.

(a) Compute $L(10^{-6})$.

[5] To evaluate an expression of log with base 10 using a calculator, look for a 'log' key. As we will discover shortly, '$\log_{10}$' is frequently simplified as 'log'.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity *I* is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

75. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\mathrm{pH} = -\log_{10}\left[\mathrm{H}^+\right]$ where $\left[\mathrm{H}^+\right]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

(a) The hygrogen ion concentration of pure water is $\left[\mathrm{H}^+\right] = 10^{-7}$. Find its pH.

(b) Find the pH of a solution with $\left[\mathrm{H}^+\right] = 6.3\times10^{-13}$.

(c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

4.2 Properties of Logarithms

Learning Objectives

- Use the definitions of common and natural logarithms in solving equations and simplifying expressions.
- Use the change of base property to evaluate logarithms.
- Solve exponential equations using logarithmic properties.
- Combine and/or expand logarithmic expressions.
- Solve basic logarithmic equations using properties of logarithms and exponential functions.

Common and Natural Logarithms

A commonly used base for logarithms is base 10. Logarithms with base 10 are referred to as **common logarithms** and we usually leave off the base when writing these logarithms, so that $\log_{10} x = \log x$. Below is an application of common logarithms.

Example 4.2.1. The state of Utah has one of the fastest growing populations in the country with an annual growth rate of about 2%. The number of years it takes for a population with the growth rate of r to double in size is $y = \dfrac{\log 2}{\log(1+r)}$. At the current rate, how many years will it take for the population of Utah to double in size? If the growth rate is reduced to 1.5%, how many years will it take for the population to double in size?

Solution. For an annual growth rate of 2%, we set $r = 0.02$ and use a calculator to determine y.

$$y = \frac{\log 2}{\log(1+0.02)} = \frac{\log 2}{\log 1.02} \approx 35.0028$$

We find that it will take about 35 years for the population of Utah to double in size. For the annual growth rate of 1.5%, we set $r = 0.015$.

$$y = \frac{\log 2}{\log(1+0.015)} = \frac{\log 2}{\log 1.015} \approx 46.5555$$

By reducing the growth rate to 1.5%, the time for the population of Utah to double in size becomes approximately 46 and one-half years.

□

Another important base is an irrational number designated by the letter 'e'. This number arises naturally in the study of Calculus and financial transactions. The choice of the letter e is by the prolific mathematician Leonhard Euler, and is used universally. The value of e is approximately 2.718281828459. The number e may be defined as the value of the exponential term $\left(1+\frac{1}{m}\right)^m$ where the value of m gets large, as seen from the following table.

m	$\left(1+\frac{1}{m}\right)^m$
1	$\left(1+\frac{1}{1}\right)^1 = 2$
100	$\left(1+\frac{1}{100}\right)^{100} = 2.70481\cdots$
10,000	$\left(1+\frac{1}{10000}\right)^{10000} = 2.71814\cdots$
1,000,000	$\left(1+\frac{1}{1000000}\right)^{1000000} = 2.71828\cdots$

Noting that $e > 1$, we have already studied the graphs and properties of exponential functions like $f(x) = e^x$. A logarithm with base e is called a **natural logarithm**. Natural logarithms are used more often than common logarithms in mathematics, and so have their own notation. We use the notation 'ln' for 'natural logarithm', so that $\ln x = \log_e x$.

Properties of Logarithms

We next introduce some properties of logarithms that we will apply in calculating values of logarithms, combining or expanding logarithmic expressions, and solving both exponential and logarithmic equations. We begin with a property that will allow us to rewrite a logarithm as a quotient of logarithms having a different base that the original logarithm.

Example 4.2.2. Show that $\log_2 3 = \dfrac{\log 3}{\log 2}$.

Solution. By definition, the logarithm $\log_2 3$ is the power of 2 that gives 3, so we must show this power is $\frac{\log 3}{\log 2}$. We let $m = \log 3$ and $n = \log 2$ and proceed with converting each equation to exponential form.

$$m = \log 3 \Rightarrow 3 = 10^m$$
$$n = \log 2 \Rightarrow 2 = 10^n \Rightarrow 2^{\frac{1}{n}} = 10$$

It follows that

$$\begin{aligned} 2^{\frac{\log 3}{\log 2}} &= 2^{\frac{m}{n}} \\ &= \left(2^{\frac{1}{n}}\right)^m \\ &= (10)^m \\ &= 3 \end{aligned}$$

Thus, $\frac{\log 3}{\log 2}$ is the power of 2 that gives 3, and we conclude that $\log_2 3 = \frac{\log 3}{\log 2}$.

□

This property can be generalized as follows.

Change of Base Property

Let a and b be positive numbers, not equal to 1, and let x be a positive number.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

The two logarithm buttons commonly found on calculators are 'LOG' and 'LN', which correspond to the common and natural logarithms, respectively. For calculation purposes, we choose the new base a to be 10 or e:

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

Example 4.2.3. Convert the expression $\log_4 5$ to base e. Then evaluate using a calculator.

Solution. Applying the change of base property with $b = 4$, $x = 5$, and $a = e$ leads us to write $\log_4 5 = \frac{\ln 5}{\ln 4}$. Evaluating with a calculator results in an approximate value: $\log_4 5 = \frac{\ln 5}{\ln 4} \approx 1.16$.

□

Our next properties are simply restatements of results from $f(x)=b^x$ and $g(x)=\log_b x$ being inverses of each other. In **Section 4.1**, we verified that these functions are inverses when $b=2$. Before moving on, we verify that $f(x)=b^x$ and $g(x)=\log_b x$ are inverses for any valid value of b.

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f(\log_b x)\\&=b^{\log_b x}\\&=x\end{aligned}$$

for any positive number x, since $\log_b x$ is the power of b that gives x.

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g(b^x)\\&=\log_b b^x\\&=x\end{aligned}$$

for any real number x, since x is the power of b that gives b^x.

The last step in each part of the verification process gives us an inverse property, as stated below.

Inverse Properties

Let b be a positive number, not equal to 1.

$$b^{\log_b x}=x\text{, for any positive number } x$$

$$\log_b b^x=x\text{, for any real number } x$$

A result of these properties is that we can think of the exponential and logarithmic functions as 'undoing' each other.

Example 4.2.4. Show that $\log_b b=1$ and that $\log_b 1=0$.

Solution. We use the property $\log_b b^x=x$ to verify each of these equations. For $\log_b b$, we note that $\log_b b=\log_b b^1$, and it follows from the definition of a logarithm that it is equal to 1. We also note that $\log_b 1$ can be written as $\log_b b^0$, which shows that $\log_b 1$ is equal to 0.

□

We continue with an example that leads to our next property of logarithms.

Example 4.2.5. Show that $\log 4^5=5\log 4$.

Solution. By definition, $\log 4^5$ is the power of 10 that gives 4^5, so we must show this power is $5\log 4$. We proceed with a simplification of $10^{5\log 4}$.

$$\begin{aligned}10^{5\log 4}&=\left(10^{\log 4}\right)^5\\&=(4)^5\\&=4^5\end{aligned}$$

We have shown that $10^{5\log 4} = 4^5$, and thus verified that $\log 4^5 = 5\log 4$.

□

This property can be generalized as follows.

Exponent Property of Logarithms

Let b be a positive number, not equal to 1, m any real number, and x a positive number.

$$\log_b x^m = m\log_b x$$

This frequently used property can be thought of as 'putting the exponent of an argument out front as the coefficient'. A simple, but useful, consequence is obtained by setting $x = b$:

$$\begin{aligned}\log_b b^m &= m\log_b b \\ &= m(1) \\ &= m\end{aligned}$$

That is, $\log_b b^m = m$.

Using Properties of Logarithms to Solve Exponential Equations

Example 4.2.6. Solve $2^x = 3$.

Solution. The x we are looking for is the power of 2 that gives 3. By definition, $x = \log_2 3$, and by the change of base property, we find $x = \log_2 3 = \dfrac{\log 3}{\log 2}$.

Another way we can solve this equation is to take the common logarithm of both sides, after which we use the exponent property of logarithms to move the exponent x in front of the logarithm.

$$\begin{aligned}2^x &= 3 \\ \log 2^x &= \log 3 \\ x\log 2 &= \log 3 \\ x &= \frac{\log 3}{\log 2}\end{aligned}$$

□

The technique of taking the (same base) logarithm of both sides may be used to solve exponential equations such as the following.

Example 4.2.7. Solve $7^{x+1} = 3^{-2x}$.

Solution. We begin by taking the natural logarithm of both sides, and then apply the exponent property of logarithms to both sides of the equation.

$$\begin{aligned} 7^{x+1} &= 3^{-2x} \\ \ln 7^{x+1} &= \ln 3^{-2x} \\ (x+1)\ln 7 &= -2x\ln 3 \end{aligned}$$

Even though the resulting equation looks complicated, keep in mind that $\ln 7$ and $\ln 3$ are just constants. The equation $(x+1)\ln 7 = -2x\ln 3$ is a linear equation and as such we gather all terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{aligned} (x+1)\ln 7 &= -2x\ln 3 \\ x\ln 7 + \ln 7 &= -2x\ln 3 \\ x\ln 7 + 2x\ln 3 &= -\ln 7 \\ x(\ln 7 + 2\ln 3) &= -\ln 7 && \text{factor out } x \\ x &= \frac{-\ln 7}{\ln 7 + 2\ln 3} \end{aligned}$$

The answer we have found is an exact solution. An approximate solution, found using a calculator, is $x \approx -0.47$.

□

In the previous example, we could have used any logarithmic base; natural logarithm is the most frequently used, but bases 3, 7, or 10 are also good choices. The final anwers may look different, but they all represent the same value. The next two examples lead us to our last two properties of logarithms.

Example 4.2.8. Show that $\ln 15 = \ln 3 + \ln 5$.

Solution. We show this relationship using the definition of logarithms. Since $\ln 15$ is the power of e that gives 15, we want to show this power is $\ln 3 + \ln 5$. Starting with $e^{\ln 3+\ln 5}$, we apply properties of exponents, along with an inverse property of logarithms.

$$\begin{aligned} e^{\ln 3+\ln 5} &= e^{\ln 3}e^{\ln 5} \\ &= (3)(5) \\ &= 15 \end{aligned}$$

Thus, $\ln 3 + \ln 5$ is the power of e that gives 15 and we have shown that $\ln 15 = \ln 3 + \ln 5$.

□

Example 4.2.9. Show that $\log\left(\frac{u}{v}\right) = \log u - \log v$, where u and v are positive numbers.

Solution. By the definition of logarithms, we need to show that the power of 10 that gives $\frac{u}{v}$ is $\log u - \log v$. We begin with $10^{\log u - \log v}$, to which we apply properties of exponents and an inverse property of logarithms.

$$\begin{aligned} 10^{\log u - \log v} &= 10^{\log u} 10^{-\log v} \\ &= \frac{10^{\log u}}{10^{\log v}} \\ &= \frac{u}{v} \end{aligned}$$

So $\log u - \log v$ is the power of 10 that gives $\frac{u}{v}$, verifying that $\log\left(\frac{u}{v}\right) = \log u - \log v$.

□

These properties can be generalized as follows.

Sum and Difference Properties of Logarithms

Let b be a positive number, not equal to 1, and let u and v be positive numbers.

$$\log_b(uv) = \log_b u + \log_b v$$

$$\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$$

These are not really two different properties, since the second can be derived from the first as follows.

$$\begin{aligned} \log_b\left(\frac{u}{v}\right) &= \log_b\left(uv^{-1}\right) \\ &= \log_b u + \log_b v^{-1} \\ &= \log_b u - \log_b v \end{aligned}$$

It may help to remember these properties in words:

The log of a product is the sum of the logs of its factors: $\log_b(uv) = \log_b u + \log_b v$.

The log of a quotient is the log of its top minus the log of its bottom: $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$.

Combining and/or Expanding Logarithmic Expressions

Example 4.2.10. Expand the following using the properties of logarithms. Write exponents as coefficients and simplify. Assume that all variables represent positive real numbers.

1. $\log_{0.1}\left(10x^2\right)$
2. $\log \sqrt[3]{\frac{100x^2}{y z^5}}$

Solution.

1. We expand $\log_{0.1}\left(10x^2\right)$ as follows.

$$\begin{aligned}\log_{0.1}\left(10x^2\right) &= \log_{0.1}10+\log_{0.1}x^2 && \text{sum property}\\ &= \log_{0.1}10+2\log_{0.1}x && \text{exponent property}\\ &= -1+2\log_{0.1}x && \text{since } (0.1)^{-1}=10\end{aligned}$$

We find that $\log_{0.1}\left(10x^2\right)$, when expanded, is equivalent to $2\log_{0.1}(x)-1$.

2. In expanding $\log\sqrt[3]{\dfrac{100x^2}{yz^5}}$, we begin by writing the cube root as the exponent $\dfrac{1}{3}$.

$$\begin{aligned}\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{\frac{1}{3}}\\ &= \frac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{exponent property}\\ &= \frac{1}{3}\left[\log\left(100x^2\right)-\log\left(yz^5\right)\right] && \text{difference property}\\ &= \frac{1}{3}\log\left(100x^2\right)-\frac{1}{3}\log\left(yz^5\right)\\ &= \frac{1}{3}\left(\log 100+\log x^2\right)-\frac{1}{3}\left(\log y+\log z^5\right) && \text{sum property}\\ &= \frac{1}{3}\log 100+\frac{1}{3}\log x^2-\frac{1}{3}\log y-\frac{1}{3}\log z^5\\ &= \frac{1}{3}\log 100+\frac{2}{3}\log x-\frac{1}{3}\log y-\frac{5}{3}\log z && \text{exponent property}\\ &= \frac{2}{3}+\frac{2}{3}\log x-\frac{1}{3}\log y-\frac{5}{3}\log z && \text{since } 10^2=100\end{aligned}$$

Finally, we have the expanded expression $\dfrac{2}{3}\log x-\dfrac{1}{3}\log y-\dfrac{5}{3}\log z+\dfrac{2}{3}$.

□

Example 4.2.11. Use the properties of logarithms to write the following as a single logarithm.

1. $2\log x-3\log(x-1)$
2. $4\ln u-\ln(5u)+\ln\left(\dfrac{1}{u}\right)$

Solution.

1. The expression $2\log x-3\log(x-1)$ contains a difference of logarithms. Before using the difference property, we must deal with the coefficients of 2 on $\log x$ and 3 on $\log(x-1)$. This can be handled using the exponent property.

$$\begin{aligned} 2\log x - 3\log(x-1) &= \log x^2 - \log(x-1)^3 && \text{exponent property} \\ &= \log\left(\frac{x^2}{(x-1)^3}\right) && \text{difference property} \end{aligned}$$

2. The expression $4\ln u - \ln(5u) + \ln\left(\frac{1}{u}\right)$ contains both a difference and a sum of logarithms. We must also deal with the coefficient of 4 on $\ln u$.

$$\begin{aligned} 4\ln u - \ln(5u) + \ln\left(\frac{1}{u}\right) &= \ln u^4 - \ln(5u) + \ln\left(\frac{1}{u}\right) && \text{exponent property} \\ &= \ln\left(\frac{u^4}{5u}\right) + \ln\left(\frac{1}{u}\right) && \text{difference property} \\ &= \ln\left(\frac{u^4}{5u}\cdot\frac{1}{u}\right) && \text{sum property} \\ &= \ln\left(\frac{u^2}{5}\right) \end{aligned}$$

□

Using Properties to Solve Basic Logarithmic Equations

Example 4.2.12. Solve $\log x - \log 2 = 1$.

Solution. We start by combining the logarithmic terms into one term.

$$\begin{aligned} \log x - \log 2 &= 1 \\ \log\left(\frac{x}{2}\right) &= 1 && \text{difference property} \\ \frac{x}{2} &= 10^1 && \text{change to exponential form} \\ x &= 20 \end{aligned}$$

To check this potential solution, we replace x with 20 in the left side of the original equation:

$$\begin{aligned} \log x - \log 2 &= \log 20 - \log 2 \\ &= \log\left(\frac{20}{2}\right) \\ &= \log 10 \\ &= 1 \end{aligned}$$

Since the right side of the original equation is 1, we have confirmed that $x = 20$ is a solution.

□

Checking potential solutions is important in solving logarithmic equations, as we see in the next example.

Example 4.2.13. Solve $\log_2 x + \log_2(x-1) = 1$.

Solution. We first combine the logarithmic terms into one term.

$$\log_2 x + \log_2(x-1) = 1$$
$$\log_2\left(x(x-1)\right) = 1 \quad \text{sum property}$$
$$x(x-1) = 2^1 \quad \text{change to exponential form}$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

We have two potential solutions: $x = -1$ and $x = 2$. We recall that the domain of a logarithm consists only of positive numbers. The potential solution $x = -1$ is not acceptable since neither $\log_2 x$ nor $\log_2(x-1)$ is defined for this x value. We refer to $x = -1$ as an extraneous solution. It is easy to check that $x = 2$ does satisfy this equation. Thus, our answer is $x = 2$.

□

In general, we will have to check potential solutions of logarithmic equations, or at least be sure the logarithmic terms in the equation are defined for those values. We will discuss this later on. For now, we wrap up this section by summarizing the properties of logarithms.

Properties of Logarithms

Let a and b be positive numbers, not equal to 1; x, u and v positive numbers; m any real number.

- $\log_b x$ is the power of b that gives x: $\log_b x = y \Leftrightarrow b^y = x$
- $\log_b 1 = 0$ and $\log_b b = 1$
- $b^{\log_b x} = x$
- $\log_b b^x = x$
- $\log_b u = \log_b v \Rightarrow u = v$
- $\log_b x = \dfrac{\log_a x}{\log_a b}$
- $\log_b x^m = m \log_b x$
- $\log_b(uv) = \log_b u + \log_b v$
- $\log_b\left(\dfrac{u}{v}\right) = \log_b u - \log_b v$

4.2 Exercises

1. What is the purpose of the change of base formula? Why is it useful when using a calculator?

2. When does an extraneous solution occur? How can an extraneous solution be recognized?

In Exercises 3 – 6, use the change of base property to convert the given expression to an expression with the indicated base.

3. $\log_7 15$ to base e

4. $\log_{14} 55.875$ to base 10

5. $\log_3 (x+2)$ to base 10

6. $\log(x^2+1)$ to base e

In Exercises 7 – 12, use the change of base property to approximate the logarithm to five decimal places.

7. $\log_3 12$

8. $\log_5 80$

9. $\log_6 72$

10. $\log_4\left(\frac{1}{10}\right)$

11. $\log_{\frac{3}{5}} 1000$

12. $\log_{\frac{2}{3}} 50$

In Exercises 13 – 30, solve the equation analytically.

13. $3^{2x}=5$

14. $5^{-x}=2$

15. $5^{x}=-2$

16. $3^{x-1}=29$

17. $9^{x-10}=1$

18. $1.005^{12x}=3$

19. $e^{-5730k}=\frac{1}{2}$

20. $3^{x-1}=2^{x}$

21. $2^{x+1}=5^{2x-1}$

22. $3^{2x+1}=7^{x-2}$

23. $3^{x-1}=\left(\frac{1}{2}\right)^{x+5}$

24. $7^{3+7x}=3^{4-2x}$

25. $\log_4(4x)-\log_4\left(\frac{x}{4}\right)=3$

26. $\log_4(x)-\log_4(3)=1$

27. $\log_5(x-4)+\log_5 x=1$

28. $\log_2(x-1)+\log_2(x-3)=3$

29. $\log_3(x-4)+\log_3(x+4)=2$

30. $\log_5(2x+1)+\log_5(x+2)=1$

In Exercises 31 – 45, expand the logarithm. Assume when necessary that all logarithmic quantities are defined.

31. $\ln(x^3y^2)$

32. $\log_2\left(\frac{128}{x^2+4}\right)$

33. $\log_5\left(\frac{z}{25}\right)^3$

34. $\log\left(1.23\times10^{37}\right)$ 35. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$ 36. $\log_c\left(\dfrac{x^2}{y^3}\right)$

37. $\log_{\sqrt{2}}\left(4x^3\right)$ 38. $\log_8\left(x^2\sqrt{x-3}\right)$ 39. $\log\left(1000x^3y^5\right)$

40. $\log_3\left(\dfrac{x^2}{81y^4}\right)$ 41. $\ln\sqrt[4]{\dfrac{xy}{ez}}$ 42. $\log_6\left(\dfrac{216}{x^3y}\right)^4$

43. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$ 44. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$ 45. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 46 – 48, expand the logarithm after factoring. Assume when necessary that all quantities represent positive real numbers.

46. $\log_5\left(x^2-25\right)$ 47. $\ln\left(\dfrac{x^3\left(x^2-4\right)}{\sqrt{x^2+4}}\right)$ 48. $\log_{\frac{1}{3}}\left(9x\left(y^3-8\right)\right)$

In Exercises 49 – 60, use the properties of logarithms to write the expression as a single logarithm.

49. $\log\left(2x^4\right)+\log\left(3x^5\right)$ 50. $\ln\left(6x^9\right)-\ln\left(3x^2\right)$ 51. $4\ln x+2\ln y$

52. $\log_2 x+\log_2 y+\log_2 z$ 53. $\log_3 x-2\log_3 y$ 54. $\dfrac{1}{2}\log_3 x-2\log_3 y-\log_3 z$

55. $2\ln x-3\ln y-4\ln z$ 56. $\log x-\dfrac{1}{3}\log z+\dfrac{1}{2}\log y$ 57. $-\dfrac{1}{3}\ln x-\dfrac{1}{3}\ln y+\dfrac{1}{3}\ln z$

58. $\log_5 x-3$ 59. $3-\log x$ 60. $\log_7 x+\log_7\left(x-3\right)-2$

61. Use the given values $\ln a=2$, $\ln b=3$ and $\ln c=5$ to evaluate the following expressions.

(a) $\ln\left(\dfrac{a^2}{b^3c^4}\right)$

(b) $\ln\sqrt{a^2b^3c^4}$

62. Provide specific values for x, y and b to show that, in general,

(a) $\log_b\left(x+y\right)\neq\log_b x+\log_b y$

(b) $\log_b\left(x-y\right)\neq\log_b x-\log_b y$

(c) $\log_b\left(\dfrac{x}{y}\right) \neq \dfrac{\log_b x}{\log_b y}$

63. Research the history of logarithms, including the origin of the word ‘logarithm’ itself. Why is the abbreviation of the natural logarithm ‘ln’ and not ‘nl’?

64. There is a scene in the movie ‘Apollo 13’ in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

4.3 Exponential Equations and Functions

Learning Objectives

- Solve exponential equations.
- Determine *x*- and *y*-intercepts of graphs of exponential functions.
- Graph exponential functions.
- Solve applications of exponential functions.

An **exponential equation** is an equation that has an exponent containing a variable. Some examples of exponential equations are $7^{x+1}=3^{-2x}$, $5\left(10^{-2x}\right)-73=0$, and $e^{2x}+3e^{x}-10=0$, the first of which was solved in **Section 4.2**. In this section, we continue solving exponential equations and graphing exponential functions.

Solving Exponential Equations

We begin with a strategy for solving exponential equations that includes methods from the previous two sections.

Solving Exponential Equations

1. Rewrite the original equation in the form $b^m=a^n$ or $b^m=y$, if possible.[6]
2. For the case $b^m=a^n$:
   a) If $b=a$, use the one-to-one property of exponential functions to reduce the equation to the new equivalent equation $m=n$.
   b) If $b\neq a$, take the logarithm of both sides[7] and use the exponent property of logarithms to get a new equivalent equation.
3. For the case $b^m=y$, if $y\leq 0$ the equation has no solution.[8] If $y>0$, take the logarithm of both sides[9] and use the exponent property of logarithms to get a new equivalent equation.
4. Solve the new equivalent equation.

[6] This strategy fails if the equation cannot be written in one of these forms.
[7] Use the same base!
[8] If $y\leq 0$, this equation has no solution since the range of $y=b^x$ is the set of positive numbers.
[9] Again, same base!

In step 3, rather than taking the logarithm of both sides of $b^m = y$, we could convert the equation to its logarithmic form $m = \log_b y$.

Reducing the equation to one of the forms stated in our strategy may require additional operations, as we will demonstrate shortly. Note also that this strategy will not generate extraneous solutions so checking answers is not required, except to verify our work. In the following four examples, the reader should become familiar with the strategy for solving exponential equations by identifying the steps in each solution.

Example 4.3.1. Solve the equation $2^{x^2+5x} = \dfrac{1}{16}$.

Solution. Since $\dfrac{1}{16} = \dfrac{1}{2^4} = 2^{-4}$, we can rewrite $2^{x^2+5x} = \dfrac{1}{16}$ as $2^{x^2+5x} = 2^{-4}$. Applying the one-to-one property, we set the exponents equal to each other to arrive at the new equation $x^2 + 5x = -4$.

$$\begin{aligned} x^2 + 5x &= -4 \\ x^2 + 5x + 4 &= 0 \\ (x+4)(x+1) &= 0 \end{aligned}$$

We set each factor equal to zero, and find solutions to be $x = -4$ and $x = -1$.

□

While this first example is similar to equations we solved in **Section 4.1**, the next example requires additional steps to write the equation in the form $b^m = b^n$.

Example 4.3.2. Solve the equation $2\left(5^{3x-1}\right) = 10\left(25^x\right)$.

Solution. To write both sides using the same base, we first divide through by 2 to get $5^{3x-1} = 5\left(25^x\right)$.

$$\begin{aligned} 5^{3x-1} &= 5\left(25^x\right) \\ 5^{3x-1} &= 5\left(5^{2x}\right) && \text{since } 25 = 5^2 \\ 5^{3x-1} &= 5^{2x+1} && \text{since } 5 = 5^1 \end{aligned}$$

By the one-to-one property, $3x - 1 = 2x + 1$, which results in the solution of $x = 2$.

□

The next example requires additional steps to write the equation in the form $b^m = y$.

Example 4.3.3. Solve the equation $5\left(10^{-2x}\right) - 73 = 0$.

Solution. We begin by isolating the exponential term.

$$\begin{aligned} 5\left(10^{-2x}\right)-73&=0\\ 5\left(10^{-2x}\right)&=73\\ 10^{-2x}&=\frac{73}{5} \end{aligned}$$

Noting that we have a base of 10, we continue by taking the common logarithm of both sides.

$$\begin{aligned} \log 10^{-2x}&=\log\left(\frac{73}{5}\right)\\ -2x\log 10&=\log\left(\frac{73}{5}\right) && \text{exponent property}\\ -2x&=\log\left(\frac{73}{5}\right) && \text{since } \log 10=1\\ x&=-\frac{1}{2}\log\left(\frac{73}{5}\right) \end{aligned}$$

The solution is $x=-\frac{1}{2}\log\left(\frac{73}{5}\right)$. If an approximate decimal value is desired, using a calculator, we get $x\approx -0.582$.

□

We note that, in the preceeding example, converting $10^{-2x}=\frac{73}{5}$ to the logarithmic form $-2x=\log\left(\frac{73}{5}\right)$ would have resulted in a quicker solution. We could have also simplified $\log 10^{-2x}$ to $-2x$ using the inverse property.

Example 4.3.4. Solve the equation $e^{2x}+3e^{x}-10=0$.

Solution. This equation cannot be written in one of the desired forms, but it is quadratic in form since $e^{2x}=\left(e^{x}\right)^{2}$. We may choose to use a substitution by letting $u=e^{x}$. Then $e^{2x}=\left(e^{x}\right)^{2}=u^{2}$.

$$\begin{aligned} e^{2x}+3e^{x}-10&=0\\ u^{2}+3u-10&=0\\ (u-2)(u+5)&=0 \end{aligned}$$

We find $u=2$ or $u=-5$. Substituting these results back into the exponential equation, we have $e^{x}=2$ or $e^{x}=-5$. The second equation has no solution since $-5\le 0$. Taking the natural logarithm of both sides of the first equation, we get

$$\begin{aligned} \ln e^{x}&=\ln 2\\ x&=\ln 2 && \text{inverse property} \end{aligned}$$

Our final, and only, solution is $x=\ln 2$.

□

Graphing Exponential Functions

Exponential functions were first introduced and defined in **Section 4.1**. Informally, we can think of exponential functions as functions that have an exponent that contains a variable. Some examples of exponential functions are $f(x)=2^{x+1}-3$, $f(x)=3^{2x+1}-1$, and $f(x)=10^{1-x^2}$, the first of which was graphed in **Section 4.1**. Before we continue with graphs of exponential functions, we note that finding intercepts is often important in graphing, and we apply our equation solving skills in finding *x*-intercepts in the following example.

Example 4.3.5. Find the *x*-intercepts of the graph of $f(x)=2^{x^2-5}-16$.

Solution. To find the *x*-intercepts, we solve $y=f(x)=0$ for *x*.

$$\begin{aligned} 2^{x^2-5}-16&=0 \\ 2^{x^2-5}&=16 \\ 2^{x^2-5}&=2^4 \\ x^2-5&=4 \quad \text{one-to-one property} \\ x^2&=9 \end{aligned}$$

We find $x=\pm 3$, for *x*-intercepts of $(-3,0)$ and $(3,0)$.

□

We return to graphing exponential functions, making use of the following strategy.

Graphing Exponential Functions

1. Find the domain. Recall that $y=b^x$ is defined for all real *x*-values.
2. Find the *x*- and *y*-intercepts, if any exist.
3. Use transformations if the graph can be obtained through shifts, reflections, and/or scalings of the graph of a function $y=b^x$.
4. Plot additional points, as needed, to identify or confirm the general shape of the graph.
5. Find the horizontal asymptote(s), if any exist.
   Recall that for $b>0$, $y=b^x \to 0$ as $x\to-\infty$ and for $0<b<1$, $y=b^x\to 0$ as $x\to\infty$.
6. Sketch a smooth curve that passes through intercepts and points, and approaches asymptote(s).

Generally, we expect the graph to have one horizontal asymptote. However, it is possible to have a function, for example a piecewise-defined function, whose graph has no horizontal asymptote or more than one horizontal asymptote.

Example 4.3.6. Sketch the graph of $f(x) = 3^{2x+1} - 1$.

Solution.

1. The domain of this function is $(-\infty, \infty)$.

2. To find the x-intercepts, we solve $y = f(x) = 0$.

$$\begin{aligned} 3^{2x+1} - 1 &= 0 \\ 3^{2x+1} &= 1 \\ 3^{2x+1} &= 3^0 \\ 2x+1 &= 0 \quad \text{one-to-one property} \end{aligned}$$

We get $x = -\frac{1}{2}$ for an x-intercept of $\left(-\frac{1}{2}, 0\right)$. For the y-intercept, setting $x = 0$, we have $f(0) = 3^{2(0)+1} - 1 = 2$. The y-intercept is the point $(0, 2)$.

3. We may use transformations of the graph of $y = 3^x$ to graph $y = f(x) = 3^{2x+1} - 1$. The change of input from x to $2x+1$ tells us the graph of $y = 3^x$ will shift to the left by one unit, and then be horizontally scaled by $\frac{1}{2}$. Subtracting 1 from the output means that the graph will shift down by one unit.

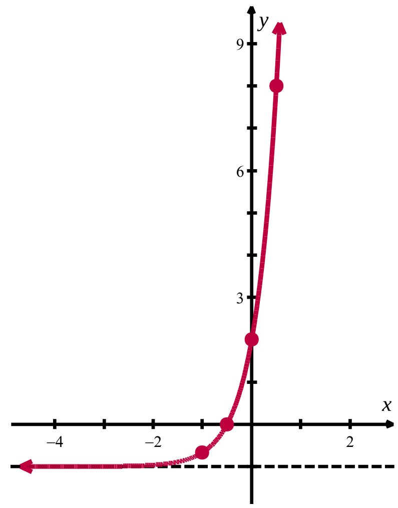
4. We may use the additional points in the table to help with the shape of the curve.

x	$y = f(x) = 3^{2x+1} - 1$	(x, y)
-1	$3^{2(-1)+1} - 1 = 3^{-1} - 1 = -\frac{2}{3}$	$\left(-1, -\frac{2}{3}\right)$
$\frac{1}{2}$	$3^{2\left(\frac{1}{2}\right)+1} - 1 = 3^2 - 1 = 8$	$\left(\frac{1}{2}, 8\right)$

5. Since we have a shift of one unit down from the graph of $y = 3^x$, whose horizontal asymptote is the line $y = 0$, the horizontal asymptote of $y = f(x) = 3^{2x+1} - 1$ is the line $y = -1$. We can also demonstrate this by evaluating f for large negative values of x.

6. We plot the intercepts, additional points, and the horizontal asymptote, then draw a smooth curve through the points that approaches the asymptote and has shape similar to $y = 3^x$.

Figure 4.3. 1



$$y = f(x) = 3^{2x+1} - 1$$

□

Example 4.3.7. Sketch the graph of $f(x) = 10^{1-x^2}$.

Solution.

1. The domain of this function is $(-\infty, \infty)$ since $1 - x^2$ is defined for any x-value.

2. To find the x-intercepts, we solve $y = f(x) = 10^{1-x^2} = 0$. This equation has no solution since $10^{1-x^2} > 0$ for all x-values; thus there is no x-intercept. To determine the y-intercept, we input $x = 0$ to get $f(0) = 10^{1-(0)^2} = 10$. The y-intercept is $(0, 10)$.

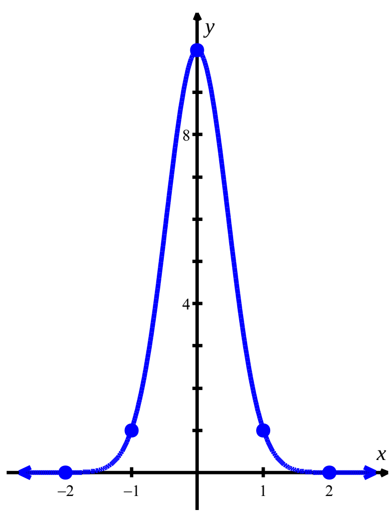
3. We cannot use transformations since our function is not a transformation of $f(x) = 10^x$ that we have seen before.

4. We identify a few more points in the following table.

x	$y = f(x) = 10^{1-x^2}$	(x, y)
-2	$10^{1-(-2)^2} = 10^{-3} = \frac{1}{1000}$	$\left(-2, \frac{1}{1000}\right)$
-1	$10^{1-(-1)^2} = 10^0 = 1$	$(-1, 1)$
1	$10^{1-(1)^2} = 10^0 = 1$	$(1, 1)$
2	$10^{1-(2)^2} = 10^{-3} = \frac{1}{1000}$	$\left(2, \frac{1}{1000}\right)$

5. By considering the values of $f(x) = 10^{1-x^2}$ as $x \to \pm\infty$, like $f(-2) = \frac{1}{1000}$ and $f(2) = \frac{1}{1000}$, we see that the horizontal asymptote is $y = 0$ (the x-axis) and that the graph approaches the horizontal asymptote both to the left and to the right.
6. We draw a smooth curve through the points we have identified, approaching the horizontal asymptote, which is the x-axis, in both directions.

Figure 4.3. 2



$y = f(x) = 10^{1-x^2}$

□

Applications of Exponential Functions

As mentioned earlier, exponential functions and equations occur frequently in everyday life. Now that we can solve exponential equations, we can also solve real life applications. Below is a more general form of an example we have already seen.

Example 4.3.8. Suppose a new car loses a fixed proportion of its value every year. Its value after t years is $V(t)=V_0(1-r)^t$ where V_0 is its initial value, and r is the proportion of yearly loss. If the car was bought initially for \$25,000 and loses 20% of its value every year, after how many years will it be worth \$10,000?

Solution. Plugging in the initial value $V_0=25000$ and proportion $r=0.2$, we have

$$\begin{aligned} V(t) &= 25000(1-0.2)^t \\ &= 25000(0.8)^t \end{aligned}$$

We want to find t when $V(t)=10000$.

$$\begin{aligned} 25000(0.8)^t &= 10000 \\ (0.8)^t &= \frac{10000}{25000} \\ 0.8^t &= 0.4 \\ \ln 0.8^t &= \ln 0.4 \\ t\ln 0.8 &= \ln 0.4 \\ t &= \frac{\ln 0.4}{\ln 0.8} \approx 4.106 \end{aligned}$$

After about 4.106 years, or about 4 years and 1.3 months, this car will be worth \$10,000.

□

We will discuss applications more thoroughly in **Section 4.5**, but for now return to logarithmic functions in **Section 4.4**.

4.3 Exercises

1. Give an example of a type of exponential equation that cannot be solved using the strategy at the beginning of this section.

2. How many intercepts will the graph of an exponential function of the form $f(x) = a^{bx+c}$, $a > 0$, $a \neq 1$, have?

In Exercises 3 – 35, solve the equation analytically.

3. $64 \cdot 4^{3x} = 16$
4. $3^{2x+1} \cdot 3^x = 243$
5. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$
6. $625 \cdot 5^{3x+3} = 125$
7. $2e^{6x} = 13$
8. $e^{r+10} - 10 = -42$
9. $2000e^{0.1t} = 4000$
10. $-8 \cdot 10^{p+7} = -24$
11. $7e^{3n-5} + 5 = -89$
12. $e^{-3k} + 6 = 44$
13. $-5e^{9x-8} - 8 = -62$
14. $-6e^{9x+8} + 2 = -74$
15. $7e^{8x+8} - 5 = -95$
16. $4e^{3x+3} - 7 = 53$
17. $8e^{-5x-2} - 4 = -90$
18. $10e^{8x+3} + 2 = 8$
19. $3e^{3-3x} + 6 = -31$
20. $500\left(1 - e^{2x}\right) = 250$
21. $30 - 6e^{-0.1x} = 20$
22. $\frac{100e^x}{e^x + 2} = 50$
23. $\frac{5000}{1 + 2e^{-3t}} = 2500$
24. $\frac{150}{1 + 29e^{-0.8t}} = 75$
25. $25\left(\frac{4}{5}\right)^x = 10$
26. $e^{2x} = 2e^x$
27. $7e^{2x} = 28e^{-6x}$
28. $e^{2x} - e^x - 132 = 0$
29. $e^{2x} - e^x - 6 = 0$
30. $e^{2x} - 3e^x - 10 = 0$
31. $e^{2x} = e^x + 6$
32. $4^x + 2^x = 12$
33. $e^x - 3e^{-x} = 2$
34. $e^x + 15e^{-x} = 8$
35. $3^x + 25 \cdot 3^{-x} = 10$

In Exercises 36 – 50, sketch the graph of $y = f(x)$. State the domain, range, *x*- and *y*-intercepts, and equation of the asymptote. Draw asymptotes as dashed lines on your graph.

36. $f(x) = 3^{\frac{x}{2}} - 2$
37. $f(x) = \left(\frac{1}{3}\right)^{-x} - 1$
38. $f(x) = 4^{-x+1}$
39. $f(x) = 5^{x-1} - 2$
40. $f(x) = -5^{x+2} + 3$
41. $f(x) = -\left(\frac{1}{2}\right)^x - 3$

42. $f(x) = e^{-x} + 2$

43. $f(x) = 2 - e^{x}$

44. $f(x) = 8 - e^{-x}$

45. $f(x) = 3\left(2^{x}\right) + 1$

46. $f(x) = 5\left(3^{-x}\right)$

47. $f(x) = 2\left(\frac{1}{3}\right)^{-x}$

48. $f(x) = 2^{x^2}$

49. $f(x) = 2^{1-x^2}$

50. $f(x) = 8 - 2^{x^2}$

In Exercises 51 – 53, determine the solution to $f(x) > 0$ for the given function. Hint: Start by finding the *x*-intercept and graphing the function over its domain.

51. $f(x) = 3^{\frac{x}{2}} - 1$

52. $f(x) = 3^{-x+2}$

53. $f(x) = 3 - e^{x}$

In Exercises 54 – 56, find the inverse of the given function.

54. $f(x) = 4^{x+5}$

55. $f(x) = e^{2x+5} + 5$

56. $f(x) = b^{-3x+7} + 4$

57. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach 20,000?

58. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where *x* is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch?

4.4 Logarithmic Equations and Functions

Learning Objectives

- Determine the domain of a logarithmic function.
- Solve logarithmic equations.
- Determine *x*- and *y*-intercepts of graphs of logarithmic functions.
- Graph logarithmic functions.
- Solve applications of logarithmic functions.

Logarithmic functions were first introduced and defined in **Section 4.1**. We identify a logarithmic function as a function involving a logarithm that has a variable in its argument. Some examples of logarithmic functions are $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$, $f(x)=\log|x-1|$ and $f(x)=\ln\left(\frac{x}{x-1}\right)$. We begin this section with determining domains of logarithmic functions.

Finding Domains of Logarithmic Functions

We recall that logarithms are only defined for positive numbers. It follows that the domain of a logarithmic function consists of those values for which its argument is a positive number.

Example 4.4.1. Find the domain of the following functions.

1. $f(x)=\log_2\left(x^2+1\right)$
2. $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$
3. $f(x)=\ln\left(\frac{x}{x+1}\right)$

Solution.

1. The logarithm $\log_2\left(x^2+1\right)$ is defined when $x^2+1>0$. This holds for all real numbers since $x^2+1\geq 1>0$ for all *x*-values. Thus, the domain of this function is all real numbers, or $(-\infty,\infty)$.

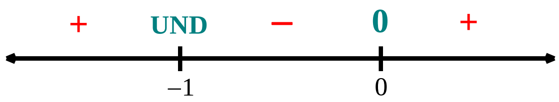
2. The function $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$ is defined when $\frac{1}{2}x+1>0$. This occurs when

$$\begin{aligned}\frac{1}{2}x &> -1\\ x &> -2\end{aligned}$$

The domain is the interval $(-2,\infty)$.

3. For the function $f(x)=\ln\left(\frac{x}{x+1}\right)$, the argument of the natural logarithm must be greater than zero, so we need to solve $\frac{x}{x+1}>0$. We let $r(x)=\frac{x}{x+1}$ represent the left side of this inequality, and note that r is undefined at $x=-1$ and is zero at $x=0$. Referring to **Section 3.4**, and evaluating the sign of r in each interval, we construct a sign diagram as follows.

Figure 4.4. 1



The domain is the set of all *x*-values less than −1 and all *x*-values greater than zero, which is the set $(-\infty,-1)\cup(0,\infty)$.

□

Solving Logarithmic Equations

A **logarithmic equation** is an equation that contains a logarithm with a variable in its argument. One example of a logarithmic equation is $\log_2 x+\log_2(x-1)=1$, which we solved in **Section 4.2**. In this section, we continue using properties of logarithms to solve logarithmic equations. In our next example, we apply the one-to-one property in solving $\ln\left(x^2-3x-1\right)=\ln(2-x)$. As we discovered in **Section 4.2**, once we have potential solutions, we must check to verify they are not extraneous.

Example 4.4.2. Solve the equation $\ln\left(x^2-3x-1\right)=\ln(2-x)$.

Solution. We can apply the one-to-one property, $\log_b u=\log_b v \Rightarrow u=v$.

$$\ln\left(x^2-3x-1\right)=\ln(2-x)$$
$$x^2-3x-1=2-x$$
$$x^2-2x-3=0$$
$$(x-3)(x+1)=0$$

We have two potential solutions: $x=-1$ and $x=3$.

Check for $x=-1$: Left Side $=\ln\left(x^2-3x-1\right)=\ln(1+3-1)=\ln 3$

Right Side $=\ln(2-x)=\ln(2+1)=\ln(3)=$ Left Side

Check for $x=3$: Left Side $=\ln\left(x^2-3x-1\right)=\ln(9-9-1)=\ln(-1)$ is not defined

Right Side $=\ln(2-x)=\ln(2-3)=\ln(-1)$ which is also not defined

So $x=3$ is not a solution. The only solution is $x=-1$.

□

Note that, in the last example, we could have stopped checking $x = 3$ after determining that the logarithm on the left side was not defined. If any logarithm in an equation is undefined for a particular value of x, that value of x is not a solution. We continue with a general strategy for solving logarithmic equations.

Solving Logarithmic Equations

1. Rewrite the original equation in the form $\log_b u = y$ or $\log_b u = \log_b v$, if possible.[10]
2. For the case $\log_b u = y$, convert the equation to its equivalent form $u = b^y$.
3. For the case $\log_b u = \log_b v$, use the one-to-one property to reduce it to the equation $u = v$.
4. Solve the new equation to find all potential solutions of the original eqution.
5. Check each potential solution in the original equation. Those that satisfy the original equation are its solutions.

The reason we must check the potential solutions is that logarithmic functions are only defined for positive numbers, not all real numbers as with exponential functions. Another way of ensuring that a potential solution is valid is to check that it is in the domain of the logarithm(s) in the original equation. Of course, this does not verify that we have not made a mistake along the way. In general, checking the potential solutions is a required part of the solution process and is not optional.

In the following two examples, the reader should become familiar with the strategy for solving logarithmic equations by identifying the steps in each solution.

Example 4.4.3. Solve the equation $\log_3(4-5x) = 1 + \log_3(x-4)$.

Solution. We move both logarithms to the same side, then use the difference property to combine them into one logarithm.

$$\begin{aligned} \log_3(4-5x) &= 1 + \log_3(x-4) \\ \log_3(4-5x) - \log_3(x-4) &= 1 \\ \log_3\left(\frac{4-5x}{x-4}\right) &= 1 \end{aligned}$$

The equation $\log_3\left(\frac{4-5x}{x-4}\right) = 1$ can be written in exponential form.

$$\frac{4-5x}{x-4} = 3^1 = 3$$

We solve the new equation by, first of all, multiplying through by $x-4$.

[10] This strategy fails if the equation cannot be written in one of these forms.

$$\begin{aligned} 4-5x &= 3(x-4) \\ 4-5x &= 3x-12 \\ -8x &= -16 \end{aligned}$$

The potential solution is $x=2$. In checking the potential solution of $x=2$, we find that $\log_3(4-5x)=\log_3(-6)$, which is not defined. Therefore, $x=2$ is not a solution. The equation does not have a solution; we say its solution set is the empty set.

□

Example 4.4.4. Solve the equation $\log(x+2)+\log(5-x)=1$.

Solution. We use the sum property to write the logarithms as one term, and proceed to write the equation in the form $\log_b u = y$.

$$\begin{aligned} \log(x+2)+\log(5-x) &= 1 \\ \log\left[(x+2)(5-x)\right] &= 1 \\ \log\left(-x^2+3x+10\right) &= 1 \end{aligned}$$

Converting the new equation into exponential form, we have $-x^2+3x+10=10^1=10$. We solve the new equation by first subtracting 10 from each side.

$$\begin{aligned} -x^2+3x &= 0 \\ x(-x+3) &= 0 \end{aligned}$$

The potential solutions are $x=0$ and $x=3$. Since the potential solution values of $x=0$ and $x=3$ result in positive arguments for $\log(x+2)$ and $\log(5-x)$, both of these values are acceptable. Of course, to be sure of our work, we could plug them in to check that they satisfy the equation. Our solutions are $x=0$ and $x=3$.

□

Example 4.4.5. Solve the equation $1+2\log_4(x+1)=2\log_2 x$.

Solution. We first gather the logs to one side to get the equation $1=2\log_2 x-2\log_4(x+1)$. Then, before we can combine the logarithms, we need a common base. Since 4 is a power of 2, we use change of base to convert $\log_4(x+1)$ to base 2.

$$\begin{aligned} \log_4(x+1) &= \frac{\log_2(x+1)}{\log_2 4} \\ &= \frac{1}{2}\log_2(x+1) \quad \text{since } \log_2 4 = 2 \end{aligned}$$

Hence, the original equation becomes

$$1 = 2\log_2 x - 2\log_4 (x+1)$$
$$1 = 2\log_2 x - 2\left(\frac{1}{2}\log_2 (x+1)\right)$$
$$1 = \log_2 \left(x^2\right) - \log_2 (x+1) \qquad \text{exponent property}$$
$$1 = \log_2 \left(\frac{x^2}{x+1}\right) \qquad \text{difference property}$$

Rewriting this equation in exponential form, we get $\frac{x^2}{x+1} = 2$ or $x^2 - 2x - 2 = 0$. Using the quadratic formula results in the potential solutions $x = 1 \pm \sqrt{3}$. Since $x = 1 - \sqrt{3}$ is negative, when substituted into the original equation, the term $2\log_2 x$ is undefined. Thus, $x = 1 - \sqrt{3}$ is an extraneous solution. The potential solution $x = 1 + \sqrt{3}$ results in positive arguments for both $2\log_4 (x+1)$ and $2\log_2 x$, so is a solution. The only solution is $x = 1 + \sqrt{3}$.

□

Graphing Logarithmic Functions

In **Section 4.1**, we graphed basic logarithmic functions and discussed their properties. We also graphed some transformations of basic logarithmic functions. Here, we state a general strategy for graphing logarithmic functions.

Graphing Logarithmic Functions

1. Find the domain. Recall that the argument of a logarithmic function must be greater than zero.
2. Find the *x*- and *y*-intercepts, if any exist.
3. Use transformations if the graph can be derived through shifts, reflections, and/or scalings of the graph of a function $y = \log_b x$.
4. Plot additional points, as needed, to identify or confirm the general shape of the graph.[11]
5. Find the vertical asymptote(s), if any exist. Recall that $y = \log_b x \to \pm\infty$ as $x \to 0^+$.
6. Sketch a smooth curve that passes through intercepts and points, and approaches asymptote(s).

Generally, if the graph has a vertical asymptote, it occurs at the end of intervals that make up the domain or at *x*-values that make the argument of the logarithm zero.

Example 4.4.6. Sketch the graph of $f(x) = \log_3 \left(\frac{1}{2}x + 1\right) + 1$.

[11] To simplify calculations, use powers of the base as the argument.

Solution.

1. In **Example 4.4.1**, we found that the domain of this function is $(-2,\infty)$.

2. To find the *x*-intercepts, we solve $y = f(x) = 0$.

$$\log_3\left(\frac{1}{2}x+1\right)+1=0$$
$$\log_3\left(\frac{1}{2}x+1\right)=-1$$
$$\frac{1}{2}x+1=3^{-1}$$
$$\frac{1}{2}x+1=\frac{1}{3}$$
$$\frac{1}{2}x=-\frac{2}{3}$$

The result is $x=-\frac{4}{3}$, for an *x*-intercept of $\left(-\frac{4}{3},0\right)$. For the *y*-intercept, inputting $x=0$, we have $f(0)=\log_3(1)+1=0+1=1$. The *y*-intercept is the point $(0,1)$.

3. We may use transformations of the graph of $y=\log_3 x$ to graph $y=f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$. The change of input from x to $\frac{1}{2}x+1$ tells us the graph of $y=\log_3 x$ will shift to the left by one unit, and then be horizontally scaled by a factor of 2. Adding 1 to the output causes the graph to shift up by one unit.

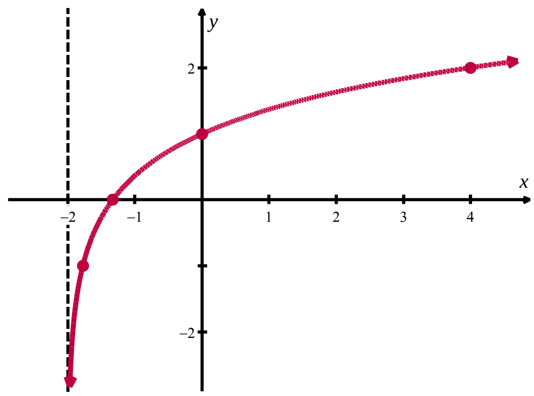
4. For additional points, we try powers of the base 3 for the value of the argument, $\frac{1}{2}x+1$, and look for points different than the *x*- and *y*-intercepts.

$\frac{1}{2}x+1$	x	$y=f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$	(x,y)
$3^{-2}=\frac{1}{9}$	$\frac{1}{2}x+1=\frac{1}{9}\Rightarrow x=-\frac{16}{9}$	$y=-2+1=-1$	$\left(-\frac{16}{9},-1\right)$
$3^{1}=3$	$\frac{1}{2}x+1=3\Rightarrow x=4$	$y=1+1=2$	$(4,2)$

5. After applying transformations to the input of $y=\log_3 x$, we see that its vertical asymptote of $x=0$ is shifted left one unit, to $x=-1$, and then multiplied by two, for a resulting vertical asymptote of $x=-2$. We can also verify this by evaluating f for values of x close to -2.

6. We plot the intercepts, additional points, and vertical asymptote, then draw a smooth curve (with shape similar to $y = \log_3 x$) through the points; the curve approaches the asymptote.

Figure 4.4. 2



$$y = f(x) = \log_3\left(\frac{1}{2}x + 1\right) + 1$$

□

Example 4.4.7. Graph the logarithmic function $f(x) = \log|x-1|$.

Solution.

1. The domain of this function consists of x-values for which $|x-1| > 0$. Since $|x-1| \geq 0$ for all x-values and $|x-1| = 0$ only for $x = 1$, the domain is the set of all real numbers except 1: $\{x \mid x \neq 1\}$ or $(-\infty, 1) \cup (1, \infty)$.

2. To find the x-intercepts, we solve $y = f(x) = \log|x-1| = 0$.

$$\begin{aligned} \log|x-1| &= 0 \\ |x-1| &= 10^0 \\ |x-1| &= 1 \\ x - 1 &= \pm 1 \end{aligned}$$

The result is $x = 0$ or $x = 2$, for x-intercepts of $(0,0)$ and $(2,0)$. The x-intercept $(0,0)$ is also the y-intercept.

3. We cannot use transformations since our function is not a transformation of $y = \log x$, due to the absolute value.

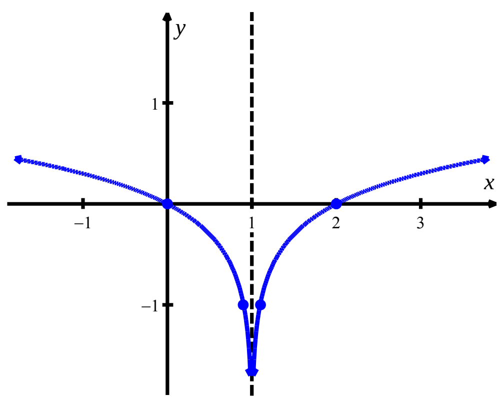
4. To add a few more points, we try powers of the base 10 for the value of the argument, $|x-1|$, looking for points that are different than the intercepts.

$\lvert x-1\rvert$	x	$y=f(x)=\log\lvert x-1\rvert$	(x,y)
$10^{-2}=\frac{1}{100}$	$\lvert x-1\rvert=\frac{1}{100}\Rightarrow x=\frac{99}{100},\frac{101}{100}$	-2	$\left(\frac{99}{100},-2\right),\left(\frac{101}{100},-2\right)$
$10^{-1}=\frac{1}{10}$	$\lvert x-1\rvert=\frac{1}{10}\Rightarrow x=\frac{9}{10},\frac{11}{10}$	-1	$\left(\frac{9}{10},-1\right),\left(\frac{11}{10},-1\right)$
$10^{1}=10$	$\lvert x-1\rvert=10\Rightarrow x=-9,11$	1	$(-9,1),(11,1)$

5. By checking x-values near 1, the end value of the domain, we have $f\left(1\pm\frac{1}{10}\right)=-1$ and $f\left(1\pm\frac{1}{100}\right)=-2$. Continuing with input values closer to $x=1$ from both sides of 1, we would find that $f(x)=\log\lvert x-1\rvert\to-\infty$, so the vertical asymptote is the line $x=1$ and the graph approaches this vertical asymptote from both sides.

6. We draw a smooth curve through the points we have identified, approaching the vertical asymptote, $x=1$, from both sides. Note that some points are not included in the following graph.

Figure 4.4. 3



$y=f(x)=\log\lvert x-1\rvert$

Notice that although we did not include the points $(-9,1)$ and $(11,1)$ in the graph, we used them as a guide on how slowly the graph is rising on both left and right sides.

□

Applications of Logarithmic Functions

Logarithmic functions and equations also occur frequently in everyday life. Now that we can solve logarithmic equations, we can also solve real life applications. Below is a problem similar to one we saw in **Section 4.2**.

Example 4.4.8. Let $P(t)$ be the population of the state of Utah, in millions, t years after 1970. In 1970, Utah had a population of about 1.06 million. Assuming a constant growth rate of 2%, the population of Utah satisfies the equation $\ln(P(t)) = \ln 1.06 + 0.02t$. Find the population of Utah as a function of t. The population of Utah in 2018 was about 3.16 million. Compare this actual population with your calculation.

Solution. We can convert our equation to exponential form and simplify.

$$\begin{aligned} \ln(P(t)) &= \ln 1.06 + 0.02t \\ P(t) &= e^{\ln 1.06 + 0.02t} \\ P(t) &= e^{\ln 1.06} e^{0.02t} \\ P(t) &= 1.06e^{0.02t} \end{aligned}$$

For the year 2018, $t = 2018 - 1970 = 48$. According to this model, the population in the year 2018 is

$$\begin{aligned} P(48) &= 1.06e^{0.02(48)} \\ &\approx 2.768 \end{aligned}$$

This is about 2.77 million, so the model has underestimated the true population of 3.16 million.

□

We note that, in reality, the population growth rate is not constant. Over the years, the population growth rate of the state of Utah has decreased from over 2% to less than 2%. We end this section with a note that many more applications, of both logarithmic and exponential functions, are coming up shortly in **Section 4.5**.

4.4 Exercises

1. What type(s) of transformation(s), if any, affect the domain of a logarithmic function?

2. What type(s) of transformation(s), if any, affect the range of a logarithmic function?

In Exercises 3 – 14, find the domain of the function.

3. $f(x) = \ln(2-x)$

4. $f(x) = \log\left(x - \frac{3}{7}\right)$

5. $h(x) = -\log(3x-4)+3$

6. $g(x) = \ln(2x+6)-5$

7. $f(x) = \log_3(15-5x)+6$

8. $f(x) = \ln(x^2+4)$

9. $f(x) = \log_7(4x+8)$

10. $f(x) = \ln(4x-20)$

11. $f(x) = \log(x^2+9x+18)$

12. $f(x) = \log\left(\frac{x+2}{x^2-1}\right)$

13. $f(x) = \log\left(\frac{x^2+9x+18}{4x-20}\right)$

14. $f(x) = \ln(7-x)+\ln(x-4)$

In Exercises 15 – 32, solve the equation analytically.

15. $\log(3x-1) = \log(4-x)$

16. $\log_2 x^3 = \log_2 x$

17. $\ln(8-x^2) = \ln(2-x)$

18. $\log_5(18-x^2) = \log_5(6-x)$

19. $\ln(x^2-99) = 0$

20. $\log\left(\frac{x}{10^{-3}}\right) = 4.7$

21. $-\log x = 5.4$

22. $10\log\left(\frac{x}{10^{-12}}\right) = 150$

23. $6-3\log_5(2x) = 0$

24. $\ln(x-2) = 1+\ln x$

25. $\log_{169}(3x+7)-\log_{169}(5x-9) = \frac{1}{2}$

26. $\ln(x+1)-\ln x = 3$

27. $2\log_7 x = \log_7 2+\log_7(x+12)$

28. $\log x-\log 2 = \log(x+8)-\log(x+2)$

29. $\log_2 x = \log_4(3x+28)$

30. $\log_3(x-2) = \log_{27}(4x+27)$

31. $\log_3 x + \log_{243}\left(x^5\right) + 3 = 0$

32. $\log_3 x = \log_{\frac{1}{3}} x + 8$

In Exercises 33 – 44, sketch the graph of $y = f(x)$. State the domain, range, *x*- and *y*-intercepts, and equation of the asymptote. Draw asymptotes as dashed lines on your graph.

33. $f(x) = \log(x+2) - 1$

34. $f(x) = -\ln(8-x)$

35. $f(x) = -10\ln\left(\dfrac{x}{10}\right)$

36. $f(x) = \log_2(x+1)$

37. $f(x) = \log_3(-x)$

38. $f(x) = \log_2(-x+3)$

39. $f(x) = -\log_3(x-2) - 4$

40. $f(x) = \ln(x-1)$

41. $f(x) = \log(x-3) + 2$

42. $f(x) = \log\left(\dfrac{1}{2}x - 1\right)$

43. $f(x) = \log_2 |x|$

44. $f(x) = \log_2 |x+1|$

In Exercises 45 – 47, determine the solution to $f(x) > 0$ for the given function. Hint: Start by finding the *x*-intercept and graphing the function over its domain.

45. $f(x) = \log(x+5) + 3$

46. $f(x) = \ln(-x+4)$

47. $f(x) = -\log_4(x+2) - 7$

In Exercises 48 – 50, find the inverse of the given function.

48. $f(x) = \log_2(x-11)$

49. $f(x) = -\ln(5-x)$

50. $f(x) = -\log_7(x-3) + 10$

51. Let $P(t)$ be the population of Arizona, in millions, *t* years after 1970. In 1970 Arizona had a population of 1.8 million. Assuming a constant growth rate of 3%, the population of Arizona satisfies the equation $\ln P(t) = \ln 1.8 + 0.03t$. Use this equation to estimate the population of Arizona in 2018. The population of Arizona in 2018 is about 7.34 million. Compare this with your estimate.

52. Let $P(t)$ be the population of the United States of America, in millions, *t* years after 1970. In 1970, the USA had a population of about 210 million. Assuming a constant growth rate of 1%, the population of the USA satisfies the equation $\ln P(t) = \ln 210 + 0.01t$. Use this equation to estimate the population of the USA in 2018. The population of the USA in 2018 is about 327 million. Compare this with your estimate.

53. Let $P(t)$ be the population of Canada, in millions, *t* years after 1970. In 1970, Canada had a population of about 21.5 million. Assuming a constant growth rate of 1.2%, the population of Canada satisfies the equation $\ln P(t) = \ln 21.5 + 0.012t$. Use this equation to estimate the population of

Canada in 2018. The population of Canada in 2018 is about 36.9 million. Compare this with your estimate.

54. Let $P(t)$ be the population of China, in millions, t years after 1970. In 1970, China had a population of about 825 million people. Assuming a constant growth rate of 0.5%, the population of China satisfies the equation $\ln P(t) = \ln 825 + 0.005t$. Use this equation to estimate the population of China in 2018. The population of China in 2018 is about 1415 million. Compare this with your estimate.

4.5 Applications of Exponential and Logarithmic Functions

Learning Objectives

- Use compound interest formulas to solve financial application problems.
- Solve application problems involving uninhibited growth and decay.
- Solve additional application problems modeled by exponential and logarithmic functions.

Exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, a calculator is often used to express our answers as decimal approximations.

Compound Interest

Let's start with an example. Suppose we invest \$10,000 in a savings account. This initial amount is called the **principal**. Assume the savings account pays 3% annual interest compounded semiannually; that is, twice a year or every six months, which is referred to as the **compounding period**. The interest is calculated at the end of each compounding period and added to the principal. After the first 6 months, the interest earned is one-half [12] of 3% of \$10,000:

$$\left(\frac{1}{2}\right)(0.03)(10000)=\left(\frac{0.03}{2}\right)(10000)=150$$

That is, the amount in our account at the end of the first 6 months is $\$10,000+\$150=\$10,150$. At the end of the second 6 months, the interest on the new amount of \$10,150 is

$$\left(\frac{0.03}{2}\right)(10150)=152.25$$

This results in an amount of $\$10,150+\$152.25=\$10,302.25$ at the end of the second 6 months. At the end of the third 6 months, interest on \$10,302.25 is

$$\left(\frac{0.03}{2}\right)(10302.25)=\$154.53$$

[12] Since interest is compounded twice a year, the six-month interest rate is one half the annual interest rate. Had interest been compounded quarterly, or four times per year, the three-month interest rate would have been one fourth of the annual interest rate.

The new amount, at the end of the third 6 months, is approximately $\$10,302.25+\$154.53=\$10,456.78$.

The pattern found in these calculations results in a compound interest formula. We let P represent the principal, r the interest rate[13], n the number of compounding periods per year, and A_k the amount in the account after the *k*th compounding. We find that A_1, the amount in our account after one compounding period, is

$$A_1 = P + \left(\frac{r}{n}\right)P = P\left(1+\frac{r}{n}\right)$$

After the second compounding period, we have

$$\begin{aligned} A_2 &= A_1 + \left(\frac{r}{n}\right)A_1 \\ &= A_1\left(1+\frac{r}{n}\right) \\ &= \left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right) \\ &= P\left(1+\frac{r}{n}\right)^2 \end{aligned}$$

After three compounding periods, the amount in our account is

$$\begin{aligned} A_3 &= A_2 + \left(\frac{r}{n}\right)A_2 \\ &= A_2\left(1+\frac{r}{n}\right) \\ &= \left[P\left(1+\frac{r}{n}\right)^2\right]\left(1+\frac{r}{n}\right) \\ &= P\left(1+\frac{r}{n}\right)^3 \end{aligned}$$

It follows that, after k compounding periods, we have $A_k = P\left(1+\dfrac{r}{n}\right)^k$. Since we compound the interest n times per year, after t years, we have $k = n \cdot t$ compoundings. We have just derived the following general formula for compound interest.

[13] When used in calculations, interest is written in decimal notation. For example, as we have shown, 3% must be written as 0.03.

Compounded Interest Formula

If a principal amount of P dollars is invested in an account earning interest at an annual rate r, compounded n times per year, the amount A in the account after t years is $A = P\left(1+\frac{r}{n}\right)^{nt}$ dollars.

Remember to input r in decimal notation!

Example 4.5.1. Suppose \$2000 is invested in an account earning 3% annual interest compounded monthly.

1. How much will be in this account after 6 years?
2. How long will it take for the value of this account to reach \$3500?
3. How long will it take for the principal of \$2000 to double in value?

Solution. The principal P is \$2000 and annual interest rate is 3%, or $r = 0.03$, compounded $n = 12$ times per year. The total accumulation in this account after t years is $A = 2000\left(1+\frac{0.03}{12}\right)^{12t}$.

1. After 6 years, setting $t = 6$, we calculate that the total accumulation is

$$\begin{aligned} A &= 2000\left(1+\frac{0.03}{12}\right)^{(12)(6)} \\ &= 2000(1.0025)^{72} \\ &\approx 2393.8969 \end{aligned}$$

There will be about \$2393.90 in the account after 6 years.[14]

2. To find the time required for the amount to reach \$3500, we set $A = 3500$ and solve for t.

$$\begin{aligned} 3500 &= 2000\left(1+\frac{0.03}{12}\right)^{12t} & & \\ \frac{3500}{2000} &= \left(1+\frac{0.03}{12}\right)^{12t} & & \text{divide by 2000} \\ 1.75 &= (1.0025)^{12t} & & \text{simplify} \\ \ln 1.75 &= \ln 1.0025^{12t} & & \text{take the natural logarithm of both sides} \\ \ln 1.75 &= (12t)\ln 1.0025 & & \text{exponent property} \\ \ln 1.75 &= (12\ln 1.0025)t & & \\ t &= \frac{\ln 1.75}{12\ln 1.0025} \approx 18.6772 & & \end{aligned}$$

It will take about 18.68 years, or about 18 years, 8 months, and 5 days, for the value of the account to reach \$3500.

[14] You may want to check out how banks do rounding. This could affect your \$2393.90.

3. To determine the time required for the original principal of \$2000 to double, we look for the value of t for which $A = 2(2000) = 4000$. We solve the following exponential equation.

$$4000 = 2000\left(1+\frac{0.03}{12}\right)^{12t}$$

$$\frac{4000}{2000} = \left(1+\frac{0.03}{12}\right)^{12t} \qquad \text{divide by 2000}$$

$$2 = (1.0025)^{12t} \qquad \text{simplify}$$

$$\ln 2 = \ln 1.0025^{12t} \qquad \text{take the natural logarithm of both sides}$$

$$\ln 2 = (12t)\ln 1.0025 \qquad \text{exponent property}$$

$$\ln 2 = (12\ln 1.0025)t$$

$$t = \frac{\ln 2}{12\ln 1.0025} \approx 23.1338$$

It will take about 23.13 years, or close to 23 years, 1 month, and 18 days, for the accumulation in this account to double in value.

□

In the next example, we know everything except the interest rate. This example may send us searching for investment opportunities but, as a side note, higher interest rates are often linked to riskier investments.

Example 4.5.2. Suppose we want to invest \$10,000, and hope to have \$15,000 in 8 years. What interest rate, compounded annually, must we seek to achieve this desired result?

Solution. We have the a principal of \$10,000 with annual compounding (once a year), so $P = 10000$ and $n = 1$. For the time period of 8 years, we use $t = 8$, and we set $A = 15000$ to give us the following equation, which we proceed to solve.

$$15000 = 10000\left(1+\frac{r}{1}\right)^{(1)(8)}$$

$$1.5 = (1+r)^8 \qquad \text{divide by 10000 and simplify}$$

$$\sqrt[8]{1.5} = 1+r \qquad \text{since } 1+r>0$$

$$r = \sqrt[8]{1.5} - 1 \approx 0.05199$$

So, we will look for an interest rate of approximately 5.2%, compounded annually.

□

The more times an investment is compounded per year, the higher the total accumulation will be. However, there is a limit to the growth as the number of compounding periods per year increases. Let's take another look at the compound interest formula $A=P\left(1+\frac{r}{n}\right)^{nt}$, replacing $\frac{r}{n}$ with $\frac{1}{\left(\frac{n}{r}\right)}$.

$$\begin{aligned} A &= P\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{nt} \\ &= P\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)rt} \qquad \text{since } n=\left(\frac{n}{r}\right)\cdot r \\ &= P\left[\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)}\right]^{rt} \end{aligned}$$

Now recall from **Section 4.2** that as m gets large, $\left(1+\frac{1}{m}\right)^{m}\to e$. In the above formula, if we think of $\frac{n}{r}$ as a variable, then as $\frac{n}{r}$ becomes large, $\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)}\to e$. In fact, since the interest rate r is a fixed value, $\frac{n}{r}$ is indeed becoming larger as n, the number of compounding periods, increases. In fact, as $n\to\infty$,

$$A=P\left(1+\frac{r}{n}\right)^{nt}\to Pe^{rt}$$

We see that the largest amount we can achieve through increasing the number of compounding periods is $A=Pe^{rt}$. Here, we say that interest is compounded continuously, and use the following formula.

Continuously Compounded Interest Formula

If a principal amount of P dollars is invested in an account earning annual interest at a rate r, compounded continuously, the total accumulation after t years is $A=Pe^{rt}$ dollars.

Example 4.5.3. Suppose you invest your newly aquired bonus of \$4000 in a retirement account that promises a minimum of 5% interest compounded continuously. How long will it take for this investment to triple in value?

Solution. For an initial amount of \$4000 and interest rate of at least 5%, we set $P = 4000$ and $r = 0.05$. The investment has tripled when the retirement account reaches three times \$4000, so we set $A = 12000$. Plugging these values into $A = Pe^{rt}$, we get

$$\begin{aligned} 12000 &= 4000e^{0.05t} \\ 3 &= e^{0.05t} && \text{divide by 4000 and simplify} \\ \ln 3 &= \ln e^{0.05t} && \text{take the natural logarithm of both sides} \\ \ln 3 &= 0.05t && \text{exponent property} \\ t &= \frac{\ln 3}{0.05} \approx 21.9722 \end{aligned}$$

It will take about 22 years for this investment to triple in value.

□

Exponential Growth and Decay

It turns out that many natural phenomena also experience exponential growth or decay like, for example, uninhibited population growth, or decay of radioactive material.

Exponential Growth or Decay

Suppose a substance grows or decays exponentially as a function of a variable t that represents time, and its amount at time t is $A(t)$. Then $A(t) = A_0 e^{kt}$ where A_0 is the initial amount and k is the relative or exponential growth/decay rate.

- If $k > 0$, A grows exponentially.
- If $k < 0$, A decays exponentially.

A few notes are in order.

- A_0 is called the initial amount since it is the value of A at time zero: $A(0) = A_0 e^0 = A_0$. We often read '*A* sub zero' as '*A* naught'.
- In this formula, we can use any positive number b, not equal to one, as the base in place of e, but doing so will change the value of k.
- This formula and the formula for continuously compounded interest, $A = Pe^{rt}$, are the same. The only differences are the use of A_0 in place of P and k in place of r. So, there is no need to memorize two different formulas.

Example 4.5.4. In order to perform anthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells grow exponentially, find a formula for the number of cells, $A(t)$, in thousands, after t days.

Solution. We start with the formula $A(t) = A_0 e^{kt}$. Then, since A is to give the number of cells in thousands, we know $A_0 = 12$, from which $A(t) = 12e^{kt}$. To complete the formula, we must determine the growth rate k. We know that after one week the number of cells has grown to five million. Since t measures days and the unit of A is thousands, this translates to $A(7) = 5000$. Noting that, additionally, $A(7) = 12e^{7k}$, we can put these two equations together and solve to find k.

$$
\begin{aligned}
12e^{7k} &= 5000 \\
e^{7k} &= \frac{5000}{12} && \text{divide by 12} \\
\ln e^{7k} &= \ln\left(\frac{5000}{12}\right) && \text{take the natural logarithm of both sides} \\
7k &= \ln\left(\frac{5000}{12}\right) && \text{inverse property} \\
k &= \frac{1}{7}\ln\left(\frac{5000}{12}\right) \approx 0.8618
\end{aligned}
$$

The number of cells, in thousands, after t days is $A(t) \approx 12e^{0.8618t}$.

□

Example 4.5.5. Iodine-131 is a commonly used radioactive isotope that helps detect how well the thyroid is functioning. Iodine-131 decays exponentially and after approximately 8 days only one-half of the original amount remains. If 5 grams of Iodine-131 is present initially, find a function that gives the amount of Iodine-131, in grams, t days later.

Solution. We begin with the formula $A(t) = A_0 e^{kt}$, with $A_0 = 5$, so that $A(t) = 5e^{kt}$. To complete the formula, we need to determine the growth rate k. Since one-half of the original amount remains after approximately 8 days, we use $A(8) = 2.5$ and note that, additionally, $A(8) = 5e^{8k}$.

$$
\begin{aligned}
5e^{8k} &= 2.5 \\
e^{8k} &= 0.5 && \text{divide by 5} \\
\ln e^{8k} &= \ln(0.5) && \text{take the natural logarithm of both sides} \\
8k &= \ln(0.5) && \text{inverse property}
\end{aligned}
$$

So $k = \dfrac{\ln(0.5)}{8} \approx -0.08664$ and the amount of Iodine-131 after t days is approximately $A(t) = 5e^{-0.08664t}$.

□

You may be wondering when the exponential model is an appropriate model. It turns out that when the amount of growth or decay in each time period is proportional to the amount at the start of the period, we have exponential growth or decay. Examples are a 2% annual interest rate, or population growth rate, and the daily 8.664% decay rate for Iodine-131. We also note that the exponential growth/decay rate k is the ratio of the rate of increase or decrease to the original size so we refer to k as the relative growth rate. The general derivation of this fact will be done in Calculus.

Another property of exponential growth or decay is that the amount of a substance will double or halve in a fixed period, known as doubling time or half-life, respectively, regardless of the initial amount present. In an example from an earlier section, we saw that the doubling time for the population of Utah, assuming 2% annual growth rate, is 35 years. According to **Example 4.5.5**, the half-life of Iodine-131 is 8 days.

Example 4.5.6. The half-life of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, how many days will it take until only 0.625 grams remain? Do not solve using the function model in the last example.

Solution. Since, after every 8 days, one-half of the starting amount remains, we can form the following table.

t days	Grams of Iodine-131
0	5
8	$\frac{5}{2} = 2.5$
16	$\frac{2.5}{2} = 1.25$
24	$\frac{1.25}{2} = 0.625$

It will take 24 days for the Iodine-131 to decay to only 0.625 grams.

□

In the last example, had we been asked to find the time until 1 gram remained, a more prudent approach would be to use the formula $A(t) = 5e^{-0.08664t}$ from **Example 4.5.5**, as follows.

$$
\begin{aligned}
1 &= 5e^{-0.08664t} && \\
0.2 &= e^{-0.08664t} && \text{divide by 5} \\
\ln 0.2 &= \ln e^{-0.08664t} && \text{take the natural logarithm of both sides} \\
\ln 0.2 &= -0.08664t && \text{inverse property} \\
t &= \frac{\ln 0.2}{-0.08664} \approx 18.576 &&
\end{aligned}
$$

We see it would take slightly over 18 and one-half days for the original 5 grams to decay to 1 gram.

Other Exponential and Logarithmic Models

Example 4.5.7. In 1619, Kepler discovered the relationship $\log T = 1.5\log d - 2.95$ between the period[15] T, in Earth years, of a planet and its average distance d, in millions of miles, from the Sun. Solve this equation for the period T. Although at that time only the six inner planets were known, this formula correctly predicted the periods of other planets unknown at that time. Find the period of the planet Uranus with the average distance of $d = 1783$ million miles from the Sun.

Solution. To solve for T, we convert this equation to exponential form.

$$
\begin{aligned}
\log T &= 1.5\log d - 2.95 && \\
T &= 10^{1.5\log d - 2.95} && \text{change to exponential form} \\
T &= 10^{1.5\log d}10^{-2.95} && \\
T &= \left(10^{\log d}\right)^{1.5}10^{-2.95} && \\
T &= (d)^{1.5}10^{-2.95} && \text{inverse property} \\
T &= 10^{-2.95}d^{1.5} &&
\end{aligned}
$$

To find the period of Uranus, we plug in $d = 1783$ and find the value of T.

$$T = 10^{-2.95}1783^{1.5} \approx 84.4748$$

The period of Uranus is about 84.5 Earth years.

□

The formula in the prior example was also used to predict the position of the asteroid belt in our solar system, giving reason to think that the asteroid belt is made of material that failed to form a planet.

Example 4.5.8. A defrosted turkey at temperature of 32° F is placed in an oven of constant temperature 350° F. The temperature of the turkey after t hours is $T(t) = 350 - 318e^{kt}$, where k is a constant. If after 2 hours the temperature of the turkey reaches 125° F, find the constant k. Determine how long it will take for the turkey to reach the safe consumption temperature of 172° F.

[15] The period of a planet is the time it takes for a planet to complete one revolution of the Sun.

Solution. To find k, we solve $T(2)=125$.

$$\begin{aligned} 125 &= 350-318e^{2k} \\ -225 &= -318e^{2k} \\ \frac{225}{318} &= e^{2k} \\ \ln\left(\frac{225}{318}\right) &= 2k \\ k &= \frac{1}{2}\ln\left(\frac{225}{318}\right) \approx -0.173 \end{aligned}$$

So $T(t) \approx 350-318e^{-0.173t}$, and we solve $T(t)=172$ to find the time it takes the turkey to reach 172° F.

$$\begin{aligned} 350-318e^{-0.173t} &= 172 \\ -318e^{-0.173t} &= -178 \\ e^{-0.173t} &= \frac{178}{318} \\ \ln e^{-0.173t} &= \ln\left(\frac{178}{318}\right) \\ -0.173t &= \ln\left(\frac{178}{318}\right) \\ t &= \frac{1}{-0.173}\ln\left(\frac{178}{318}\right) \approx 3.354 \end{aligned}$$

It will take about 3 hours and 21 minutes for the turkey to reach 172° F.

□

The model used in the above example is called Newton's Law of Heating and Cooling. We move on to one last example before ending this section, and chapter.

Example 4.5.9. The number of students, N, in hundreds, at Salt Lake Community College who have acted on the rumor 'Free popsicles at the library!', and received a free popsicle, can be modeled using the equation $N(t)=\dfrac{84}{1+2799e^{-t}}$, where $t \ge 0$ is the number of days after June 4, 2018. How many students got a free popsiscle on June 4, 2018? After how many days, will 4200 students have picked up their free popsicle?

Solution. On June 4, 2018, $t=0$ and

$$\begin{aligned} N(0) &= \frac{84}{1+2799e^{0}} \\ &= \frac{84}{2800} \\ &= \frac{3}{100} \end{aligned}$$

So the number of students who got a free popsicle on June 4, 2018, was $\frac{3}{100}$ hundreds, or $\frac{3}{100}\times 100 = 3$.

To find how many days it takes for 4200 students to get a free popsicle, we need to solve $N(t) = 42$.

$$\begin{aligned} \frac{84}{1+2799e^{-t}} &= 42 \\ 84 &= 42\left(1+2799e^{-t}\right) \\ \frac{84}{42} &= 1+2799e^{-t} \\ 2-1 &= 2799e^{-t} \\ \frac{1}{2799} &= e^{-t} \\ \ln\left(\frac{1}{2799}\right) &= \ln e^{-t} \\ -t &= \ln\left(\frac{1}{2799}\right) \end{aligned}$$

We find $t = -\ln\left(\frac{1}{2799}\right) \approx 7.937$. It takes about 8 days for 4200 students to get a free popsicle. That's a lot of popsicles!

□

Note that the model used in **Example 4.5.9** is a logistic model.

4.5 Exercises

1. What is the effect of interest on a savings account being compounded monthly versus quarterly?

2. How is continuously compounded interest related to exponential growth and decay?

In Exercises 3 – 8, find each of the following.

(a) the amount A in the account as a function of the term of the investment t in years;

(b) how much is in the account after 5 years, 10 years, 30 years and 35 years, rounding your answers to the nearest cent;

(c) how long it will take the initial investment to double, rounding your answer to the nearest year.

3. \$500 is invested in an account that offers 0.75%, compounded monthly.

4. \$500 is invested in an account that offers 0.75%, compounded continuously.

5. \$1000 is invested in an account that offers 1.25%, compounded monthly.

6. \$1000 is invested in an account that offers 1.25%, compounded continuously.

7. \$5000 is invested in an account that offers 2.125%, compounded monthly.

8. \$5000 is invested in an account that offers 2.125%, compounded continuously.

9. Look back at your answers to Exercises 3 – 8. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.

10. How much money needs to be invested now to obtain \$2000 in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.

11. How much money needs to be invested now to obtain \$5000 in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.

12. If the Annual Percentage Rate (APR) for a savings account is 0.25% compounded monthly, use the equation $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ to answer the following.

(a) For a principal of \$2000, how much is in the account after 8 years?

(b) If the original principal was \$2000 and the account now contains \$4000, how many years have passed since the original investment, assuming no other additions or withdrawals have been made?

(c) What principal should be invested so that the account balance is \$2000 in three years?

13. If the Annual Percentage Rate (APR) for a 36-month Certificate of Deposit (CD) is 2.25%, compounded monthly, use the equation $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ to answer the following.

(a) For a principal of \$2000, how much is in the account after 8 years?

(b) If the original principal was \$2000 and the account now contains \$4000, how many years have passed since the original investment, assuming no other additions or withdrawals have been made?

(c) What principal should be invested so that the account balance is \$2000 in three years?

(d) The Annual Percentage Yield is the simple[16] interest rate that returns the same amount of interest after one year as the compound interest does. Compute the APY for this investment.

14. A finance company offers a promotion on \$5000 loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount will be due?

15. Use the equation $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ to show that the time it takes for an investment to double in value does not depend on the principal *P*, but rather depends on the APR and the number of compoundings per year. Let $n=12$ and with the help of your classmates compute the doubling time for a variety of rates *r*. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[17] in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

[16] There is no compounding with simple interest.
[17] Awesome pun!

In Exercises 16 – 20, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and *k* is a constant representing the rate of decay. For each isotope:

(a) Find the decay constant *k*. Round your answer to four decimal places.

(b) Find a function that gives the amount of isotope *A* that remains after time *t*. (Keep the units of *A* and *t* the same as the given data.

(c) Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

16. Cobalt-60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

17. Phosphorus-32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

18. Chromium-51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

19. Americium-241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

20. Uranium-235, used for nuclear power, initial amount 1 kg, half-life 704 million years.

21. With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises 16 – 20 to decay does not depend on the initial amount of the substance, but rather on only the decay constant *k*. Find a formula, in terms of *k* only, to determine how long it takes for 90% of a radioactive isotope to decay.

22. The Gross Domestic Product (GDP) of the US (in billions of dollars) *t* years after the year 2000 can be modeled by

$$G(t) = 9743.77e^{0.0514t}$$

(a) Find and interpret $G(0)$.

(b) According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was \$14,369.1 billion and the 2010 GDP was \$14,657.8 billion.)

23. The diameter *D* of a tumor, in millimeters, *t* days after it is detected is given by

$$D(t) = 15e^{0.0277t}$$

(a) What was the diameter of the tumor when it was originally detected?

(b) How long until the diameter of the tumor doubles?

24. Under optimal conditions, the growth of a certain strain of *E. coli* is modeled by the Law of Uninhibited Growth $A(t) = A_0 e^{kt}$ where A_0 is the initial number of bacteria and *t* is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.

    (a) Find the growth constant *k*. Round your answer to four decimal places.

    (b) Find a function that gives the number of bacteria $A(t)$ after *t* minutes.

    (c) How long until there are 9000 bacteria? Round your answer to the nearest minute.

25. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let *t* be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $A(t) = A_0 e^{kt}$.

    (a) Find the growth constant *k*. Round your answer to four decimal places.

    (b) Find a function that gives the number of yeast (in millions) per cc $A(t)$ after *t* hours.

    (c) What is the doubling time for this strain of yeast?

26. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $A(t) = A_0 e^{kt}$ that models the number of wolves *t* years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of *k* to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

27. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $A(t) = A_0 e^{kt}$ where $A(t)$ is the number of Painesville Residents *t* years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of *k* to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563.) What could be some causes for such a vast discrepancy?

28. The population of Sasquatch in Salt Lake County is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

(a) Find and interpret $P(0)$.

(b) Find the population of Sasquatch in Salt Lake County in 2013. Round your answer to the nearest Sasquatch.

(c) When will the population of Sasquatch in Salt Lake County reach 60? Round your answer to the nearest year.

29. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

(a) Use the equation $A(t) = A_0 e^{kt}$ to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

(b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

(c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

(d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

30. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use the equation $A(t) = A_0 e^{kt}$ to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in billions of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

31. Use the equation $A(t) = A_0 e^{kt}$ to show that $k = -\frac{\ln 2}{h}$ where h is the half-life of the radioactive isotope.

CHAPTER 5
HOOKED ON CONICS

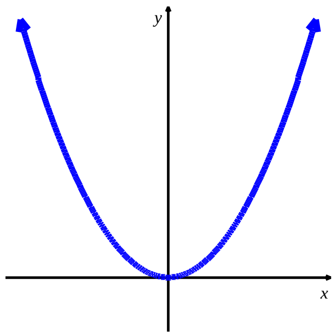


Figure 5.0. 1

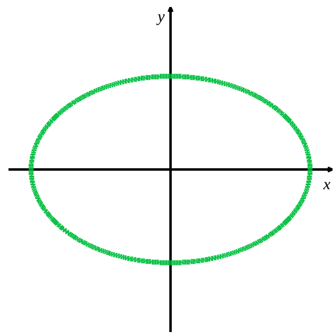


Figure 5.0. 2

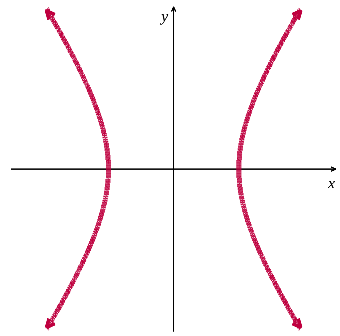


Figure 5.0. 3

Chapter Outline

Introduction

In Chapter 5, we look at circles, parabolas, ellipses, and hyperbolas, both from their geometric interpretation as conic sections and as curves defined by a set of points in the coordinate plane. By the end of this chapter, you should be able to look at an equation representing a conic and a) determine which conic is represented: circle, parabola, ellipse, or hyperbola, b) find distinguishing features, c) sketch graphs of conics, d) use knowledge of conics in application problems.

All of the conic sections are introduced in Section 5.1 through the slicing of a double-napped right circular cone by a plane to show the resulting conic sections. In Section 5.1 we are careful to distinguish between the non-degenerate cases that will be studied in this chapter and the degenerate, or trivial, cases. In the non-degenerate cases, a slice can result in a circle, an ellipse, a parabola, or a hyperbola. We will study these conic sections, one at a time, analytically in the next four sections.

In Section 5.2, circles are explored. The section begins by revisiting the definition of a circle and then introduces its standard form from which one can readily identify the center and radius of the circle. Once you are familiar with the standard form of a circle, you learn how to convert an equation of a circle not in standard form, by completing the square(s), to its standard form.

In Section 5.3, parabolas are explored. Recall from Section 2.1 that the graph of a quadratic function is a parabola and has a vertical line of symmetry. This exploration is different in two ways. First, you will be introduced to the focus and directrix of a parabola. Second, you will also work with parabolas that have horizontal lines of symmetry. As in Section 5.2, you start with the standard form of a parabola from which one can readily identify its key features and graph it. Finally, you learn to convert an equation of a parabola not in a standard form to its standard form, using completing the square.

In Section 5.4 ellipses are explored. This is likely the first time you have explored this conic section analytically, but you should be able to build from the understandings you have developed with circles and parabolas in the previous sections. The section begins by introducing the standard form for an ellipse. You will identify the vertices, major axis, foci, and graph the ellipse. Then you will do the same for equations not in standard form, using completing the square to convert to standard form.

The last section, 5.5, follows the same structure as the previous sections, but for the hyperbola. By the end of the section, you will identify the vertices, foci, and asymptotes for a hyperbola, graph the hyperbola, and manipulate an equation in non-standard form for a hyperbola to accomplish all of the above. Additionally, given certain quadratic equations in two variables, you will be able to determine the conic section each represents. You will also be able to convert a non-standard form equation of a conic to an equation in standard form, identify its key features, and sketch its graph.

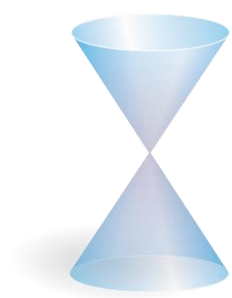
5.1 The Conic Sections

Learning Objective

- Recognize the conic sections that result from slicing a cone with a plane.

In this chapter, we study the **conic sections** – literally sections of a cone. Consider a double-napped right circular cone as seen below.[1] The point of intersection of the upper nappe with the lower nappe is called the vertex.

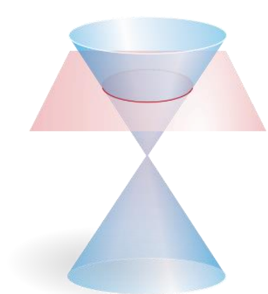
Figure 5.1. 1



Circles, Ellipses, Parabolas and Hyperbolas

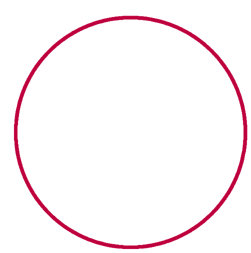
If we slice the cone with a horizontal plane, not containing the vertex, the resulting curve is a **circle**.

Figure 5.1. 2



CIRCLE

Figure 5.1. 3



[1] Graphics in this section are courtesy of Scott Nicholson. Thank you Scott!

Tilting the plane ever so slightly produces a closed curve called an **ellipse**. This will be the case for any nonhorizontal plane slicing through only the upper-half or lower-half cone.

Figure 5.1. 4

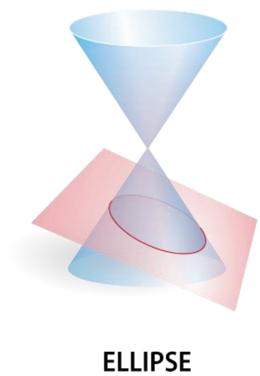
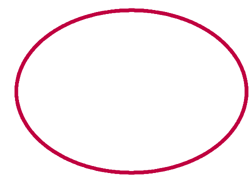


Figure 5.1. 5



If the slicing plane is parallel to the left or the right side of the cone, and does not go through the vertex, the resulting curve is a **parabola**.

Figure 5.1. 6

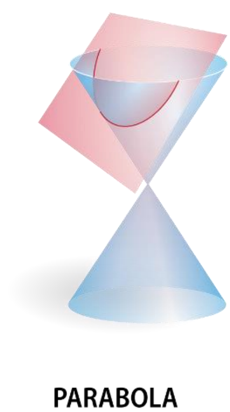
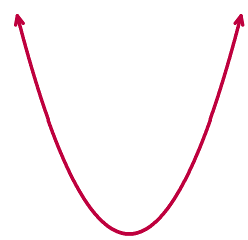


Figure 5.1. 7



If we slice the cone with a vertical plane, not through the vertex, the cross section consists of two disjoint curves, called a **hyperbola**.

Figure 5.1. 8

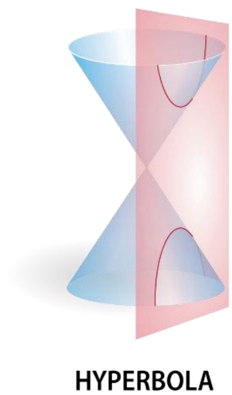
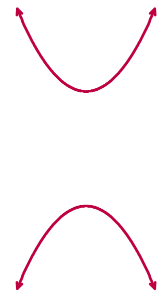


Figure 5.1. 9



The Degenerate Conic Sections

If the slicing plane contains the vertex of the cone, we get the so-called **degenerate conics**: a point, a line or two intersecting lines.

Figure 5.1. 10

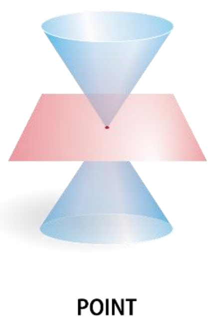


Figure 5.1. 11

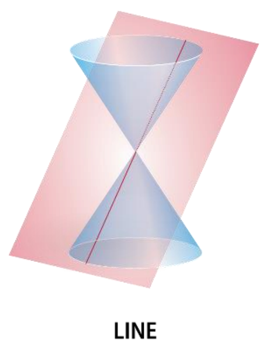
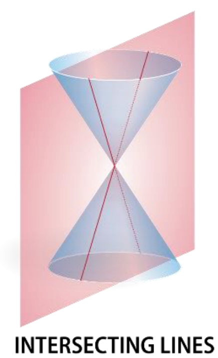


Figure 5.1. 12



In the remainder of this chapter, we will focus our discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas. As you know, points on a circle are equidistant from a point called the center. These other non-degenerate curves also have distance properties. We will use the distance properties to derive their equations.

5.2 Circles

Learning Objectives

- Define a circle in the plane.
- Write the equation of a circle in standard form.
- Graph a circle from a given equation.
- Determine the center and radius of a circle.
- Find the equation of a circle from stated properties.
- Identify the Unit Circle.

The Definition of a Circle

Recall from Geometry that a circle can be determined from a fixed point (called the center) and a positive number (called the radius) as follows.

Definition 5.1. A **circle** with center (h,k) and radius $r>0$ is the set of all points (x,y) in the plane whose distance to the point (h,k) is r.

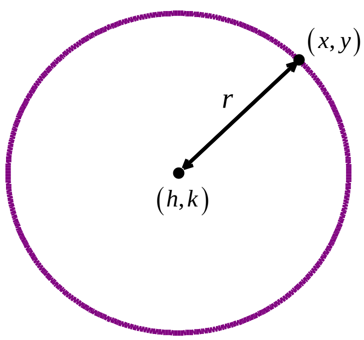


Figure 5.2. 1

By the definition, a point (x,y) is on the circle if its distance to the point (h,k) is r. We express this relationship algebraically, using the distance formula:[2]

$$r=\sqrt{(x-h)^2+(y-k)^2}$$

[2] Distance Formula: The distance d between the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is $d=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$.

By squaring both sides of this equation, and noting that $r > 0$, we get an equivalent equation that we refer to as the standard equation of a circle.

The Standard Equation of a Circle

Equation 5.1. The Standard Equation of a Circle: The equation of a circle with center (h,k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

Example 5.2.1. Write the standard equation of the circle with center $(-2,3)$ and radius 5.

Solution. For this circle, the center (h,k) is $(-2,3)$, so $h=-2$ and $k=3$. Substituting these values, along with the radius $r=5$, in the standard equation results in the following:

$$\begin{aligned}(x-h)^2 + (y-k)^2 &= r^2 \\ (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25\end{aligned}$$

□

Example 5.2.2. Find the center and radius of the circle given by the equation $(x+2)^2 + (y-1)^2 = 4$. Graph the circle.

Solution. We match the equation with the standard form of a circle, **Equation 5.1**.

$$\begin{aligned}(x-h)^2 + (y-k)^2 &= r^2 \\ (x+2)^2 + (y-1)^2 &= 4\end{aligned}$$

We see that $-h=2$, $-k=-1$, and $r^2=4$. It follows that $h=-2$ and $k=1$, for a center of $(-2,1)$. From $r^2=4$, we find $r=\pm\sqrt{4}=\pm 2$. Then, since $r>0$, the radius is $r=2$. The circle is centered at $(-2,1)$ with a radius of 2.

To graph the circle free hand, we first plot the center $(-2,1)$.[3] Since the radius is 2, we can locate four points on the circle by plotting points 2 units to the left, right, up, and down from the center. These four points can then be connected by a smooth curve, in the shape of a circle, to complete the graph. Of course, one can also graph the circle with a compass, plotting the center and using one additional point that is located 2 units from the center.

[3] Note that the center is not part of the circle. It is merely a reference point.

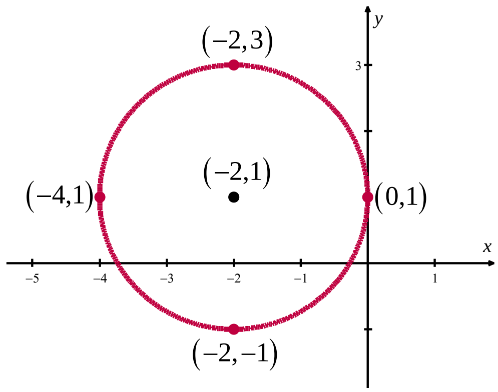


Figure 5.2. 2

□

If we expand the equation $(x+2)^2+(y-1)^2=4$ in the previous example, put all terms on one side and combine constants, we get $x^2+4x+y^2-2y+1=0$. This form is sometimes referred to as **general form**. When we are given the equation of a circle in general form, to write the equation in the standard form given in **Equation 5.1**, we complete the square in each of the variables as follows.

To Write the Equation of a Circle in Standard Form

1. Position all terms containing variables on one side of the equation, grouping terms with like variables together. Position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides by the coefficient of the square terms. (These coefficients must be equal.)

Example 5.2.3. Complete the square to find the center and radius of the circle $3x^2-6x+3y^2+4y-4=0$.

Solution.

$$
\begin{aligned}
3x^2-6x+3y^2+4y-4&=0 \\
3x^2-6x+3y^2+4y&=4 && \text{add 4 to both sides} \\
3(x^2-2x)+3\left(y^2+\frac{4}{3}y\right)&=4 && \text{factor out leading coefficients} \\
3(x^2-2x+1)+3\left(y^2+\frac{4}{3}y+\frac{4}{9}\right)&=4+3(1)+3\left(\frac{4}{9}\right) && \text{complete squares} \\
3(x-1)^2+3\left(y+\frac{2}{3}\right)^2&=\frac{25}{3} && \text{factor and simplify} \\
(x-1)^2+\left(y+\frac{2}{3}\right)^2&=\frac{25}{9} && \text{divide both sides by 3}
\end{aligned}
$$

From **Equation 5.1**, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$.

□

Before moving on, note that neither the equation $(x-3)^2+(y+1)^2=0$ nor the equation $(x-3)^2+(y+1)^2=-1$ represents a circle. To see why not, think about the points, if any, that satisfy each equation. The next example uses the midpoint formula in conjunction with the ideas presented so far in this section.

Example 5.2.4. Write the standard equation of the circle that has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting this circle using the given data yields

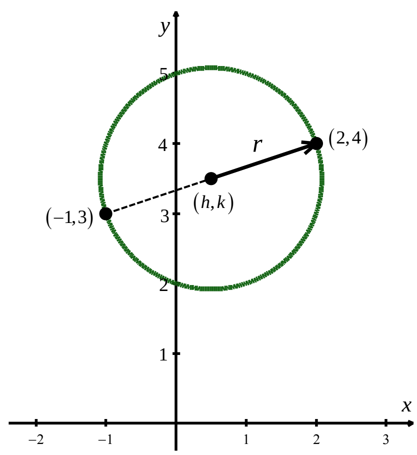


Figure 5.2. 3

Since the given points are endpoints of a diameter, we use the midpoint formula[4] to find the center (h,k).

$$\begin{aligned}(h,k)&=\left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)\\&=\left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\&=\left(\frac{1}{2},\frac{7}{2}\right)\end{aligned}$$

[4] Midpoint Formula: The midpoint M of the line segment connecting the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is $M=\left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)$.

The length of the diameter of a circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2} \\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2} \\ &= \frac{1}{2}\sqrt{3^2+1^2} \\ &= \frac{\sqrt{10}}{2} \end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4}$, the equation of the circle is $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$.

□

We close this section with a circle that is often referenced in Trigonometry.

The Unit Circle

Definition 5.2. The **Unit Circle** is the circle centered at the point $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

Example 5.2.5. Find the points on the Unit Circle that have a *y*-coordinate of $\frac{\sqrt{3}}{2}$.

Solution. We replace y with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to determine the *x*-coordinate(s).

$$\begin{aligned} x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ x^2+\frac{3}{4} &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \end{aligned}$$

We conclude that the points on the Unit Circle with a y-coordinate of $\frac{\sqrt{3}}{2}$ are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

To aid in visualization, we add the following graph:

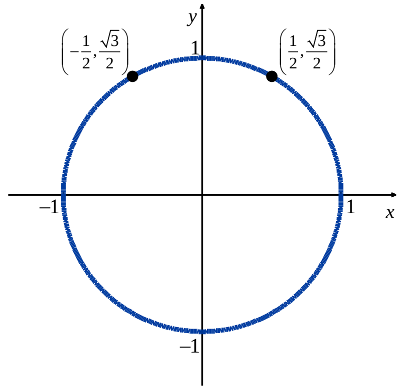


Figure 5.2. 4

□

5.2 Exercises

1. Define a circle in terms of its center.
2. In the standard equation of a circle, explain why r must be greater than 0.

In Exercises 3 – 8, write the equation in standard form if it represents a circle. If the equation does not represent a circle, explain how the equation violates the definition of a circle.

3. $x^2 - 4x + y^2 + 10y = -25$
4. $-2x^2 - 36x - 2y^2 - 112 = 0$
5. $x^2 + y^2 + 8x - 10y - 1 = 0$
6. $x^2 + y^2 + 5x - y - 1 = 0$
7. $4x^2 + 4y^2 - 24y + 36 = 0$
8. $x^2 + x + y^2 - \frac{6}{5}y = 1$

In Exercises 9 – 26, find the center and radius of the circle. Graph the circle.

9. $(x+5)^2 + (y+3)^2 = 1$
10. $(x-2)^2 + (y-3)^2 = 9$
11. $(x-4)^2 + (y+2)^2 = 16$
12. $(x+2)^2 + (y-5)^2 = 4$
13. $x^2 + (y+2)^2 = 25$
14. $(x-1)^2 + y^2 = 36$
15. $(x-1)^2 + (y-3)^2 = \frac{9}{4}$
16. $x^2 + y^2 = 64$
17. $x^2 + y^2 = 49$
18. $2x^2 + 2y^2 = 8$
19. $x^2 + y^2 + 2x + 6y + 9 = 0$
20. $x^2 + y^2 - 6x - 8y = 0$
21. $x^2 + y^2 - 4x + 10y - 7 = 0$
22. $x^2 + y^2 + 12x - 14y + 21 = 0$
23. $x^2 + y^2 + 6y + 5 = 0$
24. $x^2 + y^2 - 10y = 0$
25. $x^2 + y^2 + 4x = 0$
26. $x^2 + y^2 - 14x + 13 = 0$

In Exercises 27 – 40, put the equation of the circle that has the given properties into standard form.

27. Center $(-1,-5)$, Radius 10
28. Center $(4,-2)$, Radius 3
29. Center $\left(-3,\frac{7}{13}\right)$, Radius $\frac{1}{2}$
30. Center $(5,-9)$, Radius $\ln(8)$
31. Center $\left(-e,\sqrt{2}\right)$, Radius π
32. Center $\left(\pi,e^2\right)$, Radius $\sqrt[3]{91}$
33. Center $(3,5)$, containing the point $(-1,-2)$
34. Center $(3,6)$, containing the point $(-1,4)$
35. Center $(3,-2)$, containing the point $(3,6)$
36. Center $(6,-6)$, containing the point $(2,-3)$
37. Center $(4,4)$, containing the point $(2,2)$
38. Center $(-5,6)$, containing the point $(-2,3)$

39. Endpoints of a diameter are $(3,6)$ and $(-1,4)$

40. Endpoints of a diameter are $\left(\frac{1}{2},4\right)$ and $\left(\frac{3}{2},-1\right)$

41. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform, resulting in an overall height of 136 feet. Find an equation for the wheel assuming that its center lies on the *y*-axis and that the ground is the *x*-axis.

42. Verify that the following points lie on the Unit Circle: $(\pm 1,0)$, $(0,\pm 1)$, $\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$, and $\left(\pm\frac{\sqrt{3}}{2},\pm\frac{1}{2}\right)$.

43. The points $(-2,-2)$, $(1,1)$, $(4,-2)$, $(1,-5)$, $\left(0,-2\sqrt{2}-2\right)$, and $\left(0,2\sqrt{2}-2\right)$ lie on the circle $(x-1)^2+(y+2)^2=9$. Find three other points that lie on the circle.

5.3 Parabolas

Learning Objectives

- Define a parabola in the plane.
- Write the equation of a parabola in standard form.
- Graph a parabola from a given equation.
- Determine the vertex, focus, and directrix of a parabola.
- Find the equation of a parabola from a graph or from stated properties.
- Solve application problems modeled by parabolas.

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, is called a **parabola**. We may also define parabolas geometrically.

The Definition of a Parabola

Definition 5.3. Let F be a point in the plane and let D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

Note that the distance from a point to a line is the length of the shortest line segment connecting the point to the line; that segment is perpendicular to the line. Schematically, we have the following.

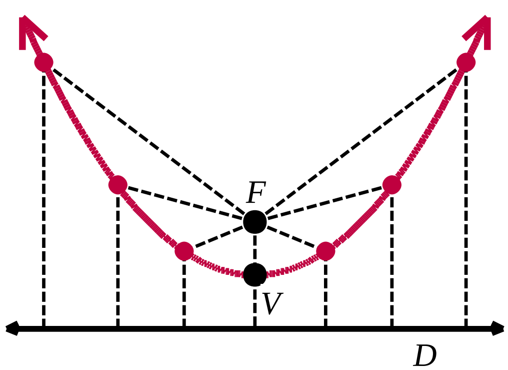


Figure 5.3. 1

Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**.

The vertex is the point on the parabola closest to the focus. The distance between the focus F and the vertex V is called the **focal length**.

The Equation of a Vertical Parabola with Vertex $(0,0)$

We want to use only the definition of parabola to derive the equation of a vertical[5] parabola. Let p denote the directed[6] distance from the vertex to the focus, which is the distance from the vertex to the directrix if the vertex is above the directrix. For simplicity, assume that the vertex is $(0,0)$ and that the parabola opens upward. Hence, the focus is $(0, p)$ and the directrix is the line $y=-p$.

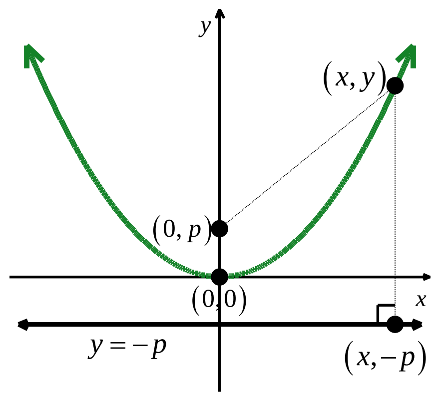


Figure 5.3. 2

From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x,-p)$ to (x, y). Using the distance formula[7], we get the following:

$$\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} \\
x^2+(y-p)^2 &= (y+p)^2 && \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 && \text{expand quantities} \\
x^2 &= 4py && \text{gather like terms}
\end{aligned}$$

Solving for y yields $y=\dfrac{x^2}{4p}$, which is a quadratic function of the form $y=ax^2$, with $a=\dfrac{1}{4p}$ and vertex $(0,0)$.

---

[5] A ‘vertical’ parabola opens either upward or downward.
[6] We’ll talk more about what ‘directed’ means later.
[7] Distance Formula: The distance d between the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is $d=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downward. In the equation $y = \frac{x^2}{4p}$, this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

Equation 5.2. The Standard Equation of a Vertical Parabola with Vertex (0,0): Let p be a non-zero real number. The equation of a vertical parabola with vertex $(0,0)$ and focal length $|p|$ is

$$x^2 = 4py$$

If $p > 0$, the parabola opens upward; if $p < 0$, the parabola opens downward.

Notice that in the standard equation of a parabola, above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle since both variables are squared in the equation of a circle.

Example 5.3.1. Graph the parabola given by the equation $x^2 = -6y$. Find the focus and directrix.

Solution. We recognize this as the form given in **Equation 5.2**. We see that $4p = -6$, so $p = -\frac{3}{2}$.

Since $p < 0$, the focus will be below the vertex and the parabola will open downward.

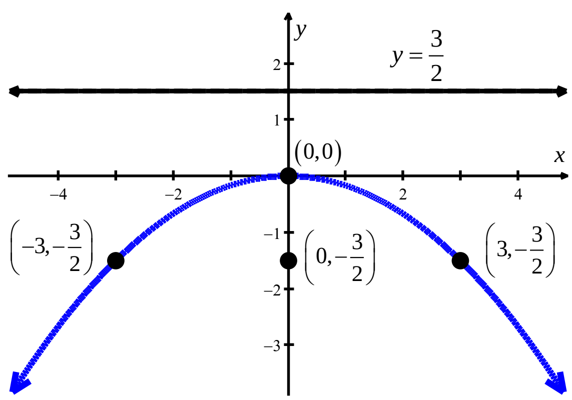


Figure 5.3. 3

The distance from the vertex to the focus is $|p| = \frac{3}{2}$, which means the focus is $\frac{3}{2}$ units below the vertex. Thus, we find the focus at $\left(0,-\frac{3}{2}\right)$. The directrix, then, is $\frac{3}{2}$ units above the vertex, so it is the line $y = \frac{3}{2}$. To find a couple of additional points on the graph, we find the points with the same *y*-coordinate as the focus. (We will see shortly why this choice was made.) Setting $y=-\frac{3}{2}$ in the equation $x^2=-6y$ results in $x^2=-6\left(-\frac{3}{2}\right)=9$, or $x=\pm 3$. So the points $\left(-3,-\frac{3}{2}\right)$ and $\left(3,-\frac{3}{2}\right)$ are on the graph.

□

Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. In order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum. The **latus rectum** of a parabola is the line segment through the focus and parallel to the directrix whose endpoints are on the parabola. Graphically, we have the following:

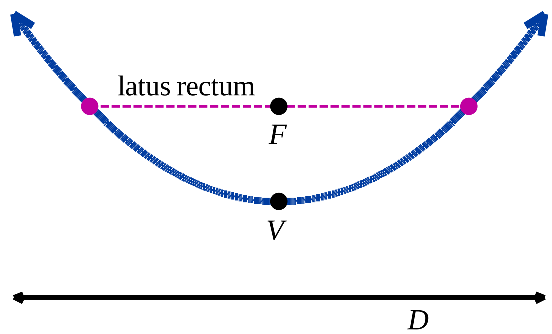


Figure 5.3. 4

The length of the latus rectum is called the **focal diameter**. In **Example 5.3.1**, the latus rectum is the line segment connecting the points $\left(-3,-\frac{3}{2}\right)$ and $\left(3,-\frac{3}{2}\right)$, and the focal diameter is 6 units. In general, the focal diameter of the parabola $x^2=4py$ is $|4p|$. To generate the graph of a vertical parabola quickly, plot points $|2p|$ units to the left and right of the focus. These are the endpoints of the latus rectum.

The following diagram summarizes the key features of the parabola.

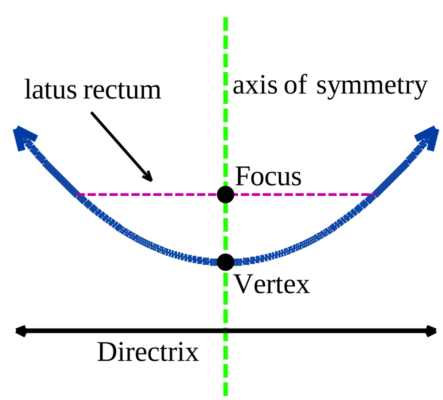


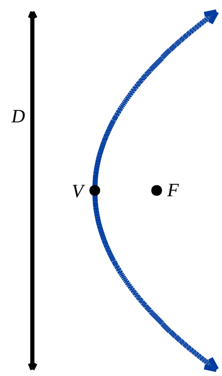
Figure 5.3. 5

We learned about the parabola's vertex and axis of symmetry while studying quadratic functions. Notice that the **axis of symmetry** passes through both the focus and the vertex and is perpendicular to the directrix.

The Equation of a Horizontal Parabola with Vertex $(0,0)$

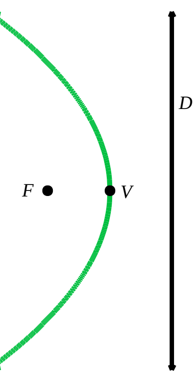
If we interchange the roles of x and y, we can produce horizontal parabolas: parabolas that open to the right or to the left. The directrix of such a parabola is a vertical line and the focus lies either to the right or to the left of the vertex, as seen below.

Figure 5.3. 6



Parabola opening to the right

Figure 5.3. 7



Parabola opening to the left

> **Equation 5.3. The Standard Equation of a Horizontal Parabola with Vertex (0,0):** Let p be a non-zero real number. The equation of a horizontal parabola with vertex $(0,0)$ and focal length $|p|$ is
>
> $$y^2 = 4px$$
>
> If $p > 0$, the parabola opens to the right; if $p < 0$, the parabola opens to the left.

The focal diameter of a horizontal parabola, like a vertical parabola, is $|4p|$. However, for a horizontal parabola, the endpoints of the latus rectum are $|2p|$ units above and below the focus.

Example 5.3.2. Graph the parabola given by the equation $y^2 = 12x$. Find the focus and directrix and the endpoints of the latus rectum.

Solution. We recognize this as the form given in **Equation 5.3**. We see that $4p = 12$, so $p = 3$. Since $p > 0$, the focus will be to the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p| = 3$, which means the focus is 3 units to the right of the vertex. If we start at the vertex, $(0,0)$, and move 3 units to the right, we arrive at the focus, $(3,0)$. The directrix, then, is 3 units to the left of the vertex, and is the vertical line $x = -3$. Since the focal diameter is $|4p| = 12$, there are two points, one 6 units above the focus and the other 6 units below the focus, that are on the parabola. These points, $(3,6)$ and $(3,-6)$, are the endpoints of the latus rectum.

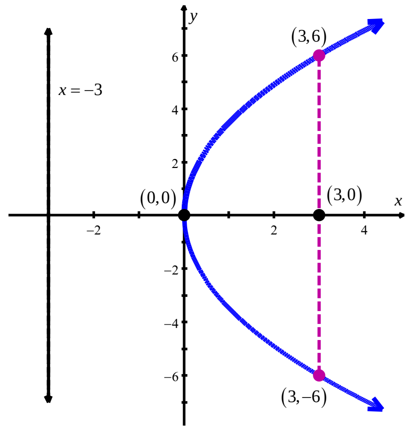


Figure 5.3. 8

□

The Equation of a Parabola with Vertex (h,k)

If we choose to place the vertex at an arbitrary point (h,k), we arrive at the following formulas using either transformations or re-deriving the formula from **Definition 5.3**.

Equation 5.4. The Standard Equation of a Parabola with Vertex (*h*, *k*): Let p be a non-zero real number.

- The equation of a vertical parabola with vertex (h,k) and focal length $|p|$ is

  $$(x-h)^2 = 4p(y-k)$$

  If $p > 0$, the parabola opens upward; if $p < 0$, the parabola opens downward.

- The equation of a horizontal parabola with vertex (h,k) and focal length $|p|$ is

  $$(y-k)^2 = 4p(x-h)$$

  If $p > 0$, the parabola opens to the right; if $p < 0$, the parabola opens to the left.

As before, the focal diameter is $|4p|$ and the endpoints of the latus rectum are $|2p|$ units from the focus.

Example 5.3.3. Graph the parabola given by the equation $(x+1)^2 = 8(y-3)$. Find the vertex, focus, directrix, and the length of the latus rectum.

Solution. This is a vertical parabola of the form $(x-h)^2 = 4p(y-k)$. Here, $x-h$ is $x+1$ so that $h=-1$, and $y-k$ is $y-3$, from which $k=3$. Hence, the vertex is $(-1,3)$. We also see that $4p=8$, so $p=2$. Since $p>0$, the focus will be above the vertex and the parabola will open upward.

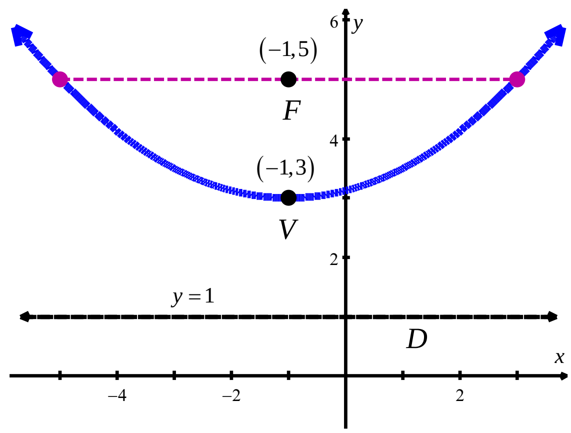


Figure 5.3. 9

The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units above the vertex. From $(-1,3)$, we move up 2 units and find the focus at $(-1,5)$. The directrix is 2 units below the vertex, so it is the line $y=1$. We see that the length of the latus rectum, also the focal diameter, is $|4p| = 8$. Thus, there are points on the parabola that are 4 units to the left and 4 units to the right of the focus. These points, $(-5,5)$ and $(3,5)$, are the endpoints of the latus rectum.

□

Example 5.3.4. Find the standard form of the parabola with focus $(2,1)$ and directrix $y=-4$.

Solution. Sketching the data yields the location of the vertex.

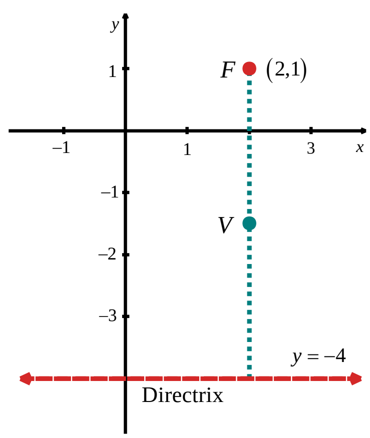


Figure 5.3. 10

From the diagram, we deduce that the parabola opens upward. (Take a moment to think about it if you do not see this immediately.) Hence, the vertex lies below the focus and has an *x*-coordinate of 2. To find the *y*-coordinate, we note that the vertex lies halfway between the focus and the directrix, and that the distance from the focus to the directrix is $1-(-4)=5$. Thus, the the vertex lies $\frac{5}{2}$ units below the focus. Starting at $(2,1)$ and moving down $\frac{5}{2}$ units leaves us at $\left(2,-\frac{3}{2}\right)$, which is the vertex. Since the parabola opens upward, we know *p* is positive. Thus, $p=\frac{5}{2}$. Plugging all of this data into $(x-h)^2=4p(y-k)$ gives us

$$(x-2)^2=4\left(\frac{5}{2}\right)\left(y-\left(-\frac{3}{2}\right)\right)$$
$$(x-2)^2=10\left(y+\frac{3}{2}\right)$$

□

As with circles, not all parabolas will come to us in the forms of **Equation 5.4**. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into the standard form of a parabola using the following steps.

To Write the Equation of a Parabola in Standard Form

1. Position all terms containing the variable that is squared on the left side of the equation and all other terms on the right side.
2. Complete the square on the left side, if necessary, and then divide both sides of the equation by the coefficient of the square term.
3. On the right side of the equation, factor out the coefficient of the variable from all terms.

Example 5.3.5. Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus and directrix.

Solution. We need a perfect square (in this case, for terms containing y) on the left side of the equation. After completing the square, we simplify and factor out the coefficient of the non-squared variable (in this case, x) on the right side.

$$\begin{aligned} y^2+4y+8x&=4 \\ y^2+4y&=-8x+4 \\ y^2+4y+4&=-8x+4+4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2&=-8x+8 \quad \text{factor} \\ (y+2)^2&=-8(x-1) \end{aligned}$$

Now that the equation is in the form given in **Equation 5.4** for a horizontal parabola, we see that $x-h$ is $x-1$, so that $h=1$, and $y-k$ is $y+2$, from which $k=-2$. Hence, the vertex is $(1,-2)$. We also see that $4p=-8$ so that $p=-2$.

Since $p<0$, the focus will be left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p|=2$, which means the focus is 2 units to the left of the vertex. If we start at $(1,-2)$ and move left 2 units, we arrive at the focus, $(-1,-2)$. The directrix is 2 units to the right of the vertex. If we move right 2 units from $(1,-2)$, we are on the vertical line $x=3$, so the directrix is $x=3$. The focal diameter is $|4p|=8$, from which we know that the parabola is 8 units wide at the focus and that there are points on the parabola that are 4 units above and 4 units below the focus.

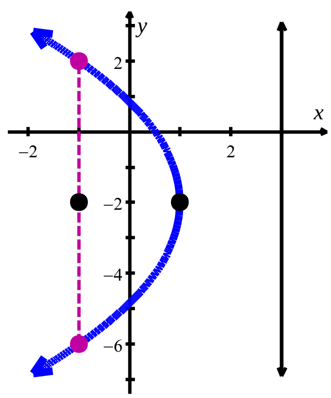


Figure 5.3. 11

□

Applications of Parabolas

In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its reflective property, which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid**, a parabola revolved about its axis of symmetry, as depicted in the following illustration.

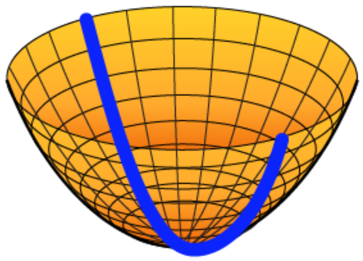


Figure 5.3. 12

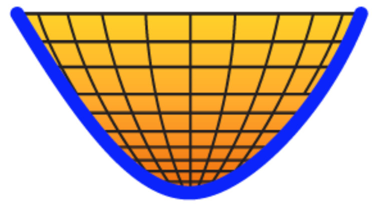


Figure 5.3. 13

Every cross section through the axis of symmetry of the generating parabola is also a parabola with the same focus. To see why this is important, imagine the dotted lines below as electromagnetic waves heading toward a parabolic dish. It turns out that the waves parallel to the axis of symmetry reflect off the parabola and concentrate at the focus, which then becomes the optimal place for the receiver.

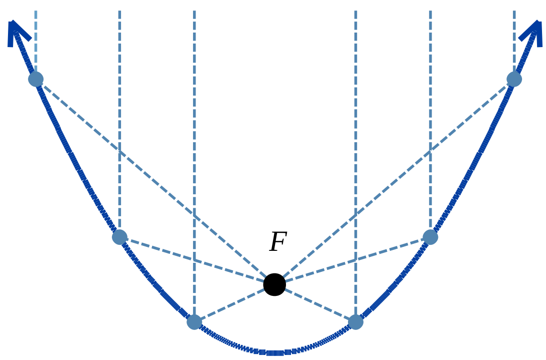


Figure 5.3. 14

If, on the other hand, we imagine the dotted lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case of a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.

Example 5.3.6. A satellite dish is to be constructed in the shape of a paraboloid. If the receiver placed at the focus is located 2 feet above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we will assume the vertex is $(0,0)$ and the parabola opens upward. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 feet above the vertex, we know $p = 2$ so that $x^2 = 8y$.

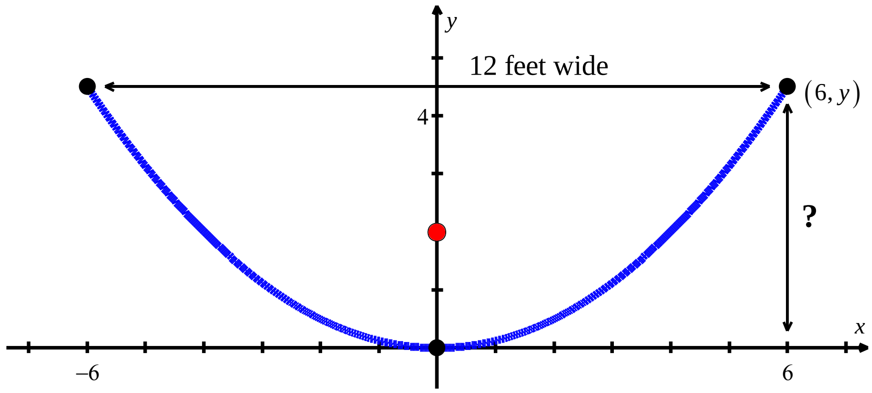


Figure 5.3. 15

With the dish having a width of 12 feet, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields

$$6^2 = 8y$$

$$y = \frac{36}{8} = 4.5$$

Hence, the dish will be 4.5 feet deep.

□

Parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors have a unique reflecting property. When rays of light parallel to the parbola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. (See the diagram preceding **Example 5.3.6**.) Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient, solar products such as solar cookers, solar heaters, and even travel-sized fire starters.

Example 5.3.7. A cross-section of a design for a travel-sized solar fire starter is shown below.

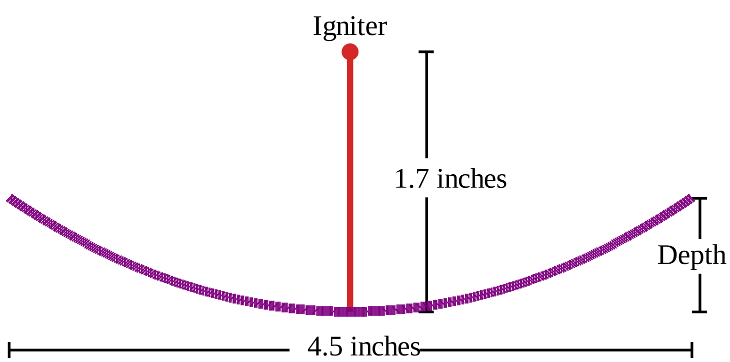


Figure 5.3. 16

The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds. Use the dimensions provided in the figure to determine the depth of the fire starter.

Solution. We begin by finding an equation for the parabola that models the fire starter. We assume that the vertex of the parabolic mirror is the origin of the coordinate plane and that the parabola has the standard form $x^2 = 4py$, where $p > 0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p = 1.7$, which we substitute into $x^2 = 4py$ to find an equation for the parabola.

$$x^2 = 4(1.7)y$$
$$x^2 = 6.8y$$

The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation to find the depth of the dish.

$$x^2 = 6.8y$$
$$(2.25)^2 = 6.8y$$
$$y \approx 0.74$$

The dish is approximately 0.74 inches deep.

□

5.3 Exercises

1. Define a parabola in terms of its focus and directrix.
2. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about the graph of the parabola?
3. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about the graph of the parabola?
4. As the graph of a parabola becomes wider, what happens to the distance between the focus and the directrix?

In Exercises 5 – 8, write the equation in standard form if it represents a parabola. If the equation does not represent a parabola, explain how the equation violates the definition of a parabola.

5. $y^2 = 4 - x^2$
6. $y = 4x^2$
7. $y^2 + 12x - 6y - 51 = 0$
8. $3x^2 - 6y^2 = 12$

In Exercises 9 – 16, find the vertex, the focus, and the directrix of the parabola. Graph the parabola. Include the endpoints of the latus rectum in your sketch.

9. $(x-3)^2 = -16y$
10. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
11. $(y-2)^2 = -12(x+3)$
12. $(y+4)^2 = 4x$
13. $(x-1)^2 = 4(y+3)$
14. $(x+2)^2 = -20(y-5)$
15. $(y-4)^2 = 18(x-2)$
16. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 17 – 22, put the equation of the parabola in standard form. Find the vertex, the focus, and the directrix. Graph the parabola.

17. $y^2 - 10y - 27x + 133 = 0$
18. $25x^2 + 20x + 5y - 1 = 0$
19. $x^2 + 2x - 8y + 49 = 0$
20. $2y^2 + 4y + x - 8 = 0$
21. $x^2 - 10x + 12y + 1 = 0$
22. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

In Exercises 23 – 31, find the standard form of the equation of the parabola that has the given properties.

23. Directrix $y = 4$, Focus $(0,-4)$

24. Directrix $x = 4$, Focus $(-4, 0)$

25. Vertex $(-2, 3)$, Directrix $x = -\frac{7}{2}$

26. Vertex $(1, 2)$, Focus $\left(1, \frac{1}{3}\right)$

27. Vertex $(0, 0)$ with endpoints of the latus rectum $(2, 1)$ and $(-2, 1)$

28. Vertex $(0, 0)$ with endpoints of the latus rectum $(-2, 4)$ and $(-2, -4)$

29. Vertex $(1, 2)$ with endpoints of the latus rectum $(-5, 5)$ and $(7, 5)$

30. Vertex $(-3, -1)$ with endpoints of the latus rectum $(0, 5)$ and $(0, -7)$

31. Vertex $(-8, -9)$, containing the points $(0, 0)$ and $(-16, 0)$

In Exercises 32 – 36, given the graph of the parabola, determine its equation.

32.

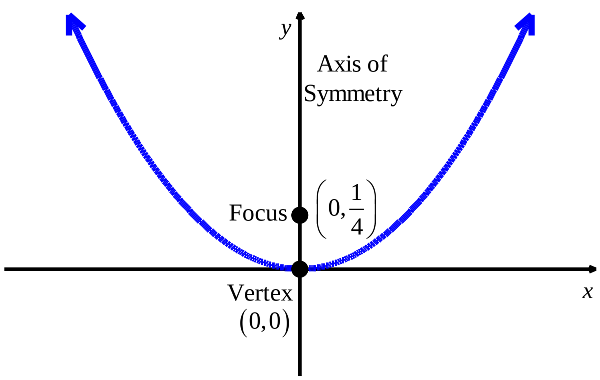


Figure Ex5.3. 1

33.

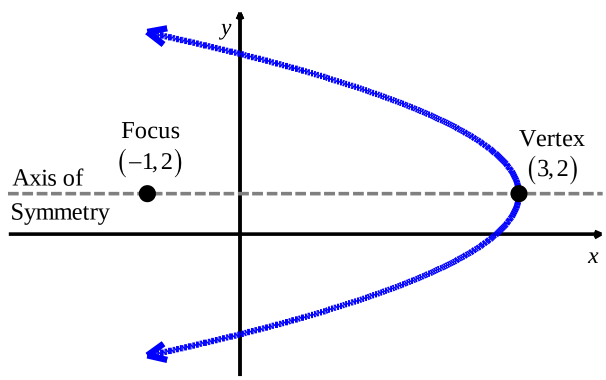


Figure Ex5.3. 2

34.

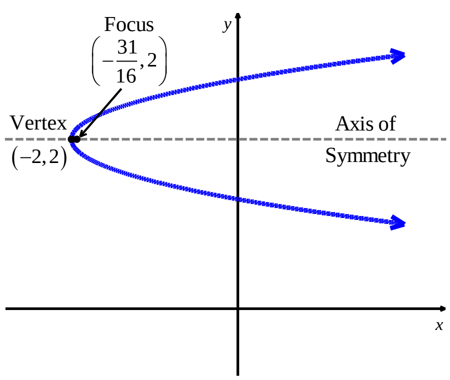


Figure Ex5.3. 3

35.

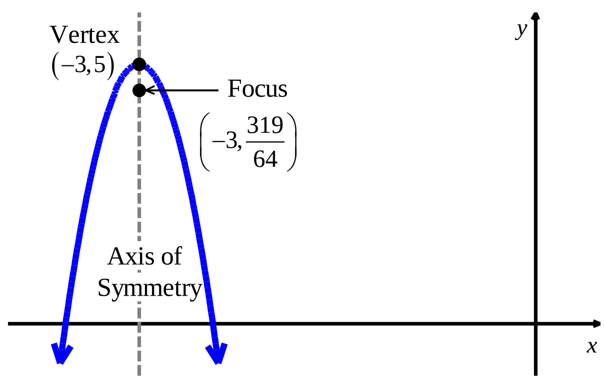


Figure Ex5.3. 4

36.

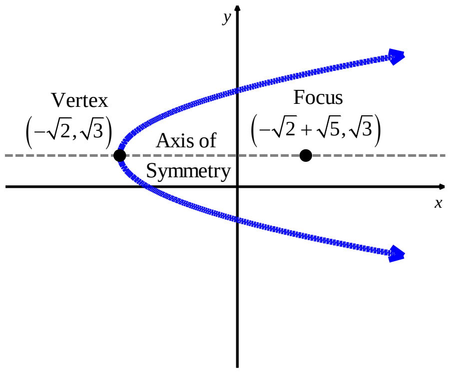


Figure Ex5.3. 5

37. The mirror in Carl's flashlight is a paraboloid. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?

38. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape. If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?

39. A parabolic arch is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

40. A satellite dish is shaped like a paraboloid. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

41. A searchlight is shaped like a paraboloid. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

42. An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Placing the vertex at the point $(0,20)$, find the equation of the parabola and determine the height of the arch 40 feet from the center.

43. Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1,600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker", which is placed 320 mm from the base. Find the depth of the cooker.

44. The points $\left(-\frac{3}{4},\frac{1}{2}\right)$, $\left(-\frac{7}{10},1\right)$, $\left(\frac{1}{2},3\right)$, $\left(0,\frac{-\sqrt{15}+1}{2}\right)$, and $\left(0,\frac{\sqrt{15}+1}{2}\right)$ lie on the parabola $\left(y-\frac{1}{2}\right)^2=5\left(x+\frac{3}{4}\right)$. Find three other points that lie on the parabola.

5.4 Ellipses

Learning Objectives

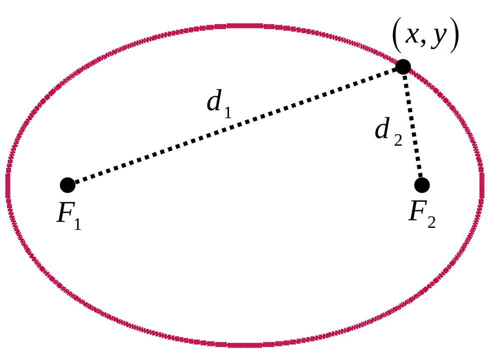
- Define an ellipse in the plane.
- Write the equation of an ellipse in standard form.
- Graph an ellipse from a given equation.
- Determine the center, vertices, foci, and eccentricity of an ellipse.
- Find the equation of an ellipse from a graph or from stated properties.
- Solve application problems modeled by ellipses.

In the definition of a circle, **Definition 5.1**, a circle is the set of points that are the same distance from a fixed point. For our next conic section, the ellipse, we consider a set of points that satisfy a similar but different property.

The Definition of an Ellipse

Definition 5.4. Given two distinct points[8] F_1 and F_2 in the plane and a number d that is larger than the distance between F_1 and F_2, an **ellipse** is the set of all points (x, y) in the plane whose sum of distances from F_1 and F_2 is d. The points F_1 and F_2 are called the **foci**[9] of the ellipse.

Figure 5.4. 1



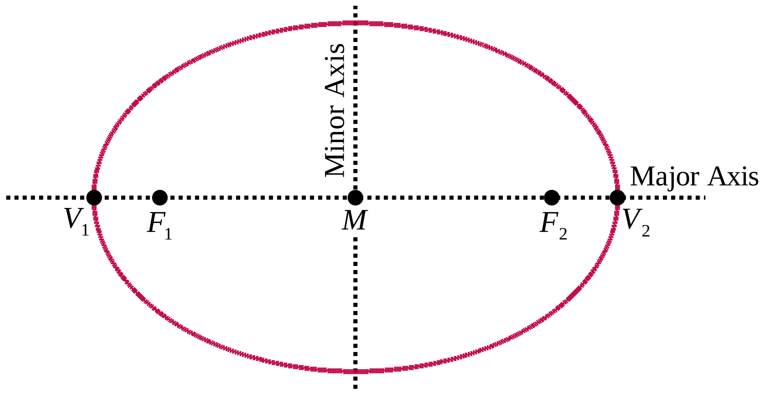
$d_1 + d_2 = d$ for all points (x, y) on the ellipse

[8] Some textbooks do not require F_1 and F_2 to be distinct points; in this case a circle is considered to be an ellipse.
[9] 'Foci' is the plural of 'focus'.

We can draw an ellipse by taking a length of string and anchoring its ends to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of an ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line through the foci and center of the ellipse. The **minor axis** of the ellipse is the line through the center and perpendicular to the major axis. The **vertices** of an ellipse are the points at which the major axis intersects the ellipse.

Figure 5.4. 2



An ellipse with center M , foci F_1 and F_2 , vertices V_1 and V_2

Note that

- In addition to being the midpoint of the foci, the center is also the midpoint of the vertices since

$$V_1F_1 + V_1F_2 = V_2F_1 + V_2F_2 \qquad \text{definition of an ellipse}$$
$$V_1F_1 + V_1F_1 + F_1F_2 = V_2F_2 + F_2F_1 + V_2F_2$$
$$V_1F_1 + V_1F_1 = V_2F_2 + V_2F_2$$
$$V_1F_1 = V_2F_2$$
$$V_1F_1 + F_1M = V_2F_2 + F_2M$$
$$V_1M = V_2M \qquad M \text{ is center of foci so } F_1M = F_2M$$

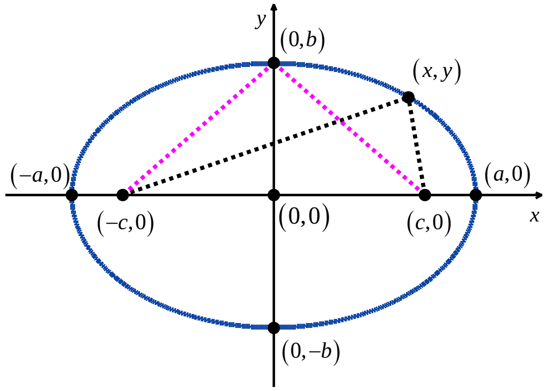
- As will be discussed later, the segment of the major axis within (bounded by) the ellipse is longer than the segment of the minor axis within (bounded by) the ellipse. This will always be the case, whether the major axis is horizontal or vertical.

We move on to equations of ellipses in the coordinate plane.

The Equation of an Ellipse with Center at $(0,0)$

In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$; its major axis is the *x*-axis; its foci are $(c,0)$ and $(-c,0)$; its vertices are $(a,0)$ and $(-a,0)$. We label the *y*-intercepts of the ellipse as $(0,b)$ and $(0,-b)$. We assume a, b, and c are all positive numbers.

Figure 5.4. 3



Since $(a,0)$ is on the ellipse, it must satisfy the conditions of **Definition 5.4**. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the *x*-axis, we get the following:

$$\begin{aligned}\left[\text{distance from } (-c,0) \text{ to } (a,0)\right]+\left[\text{distance from } (c,0) \text{ to } (a,0)\right]&=d\\ (a+c)+(a-c)&=d\\ 2a&=d\end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is the length of the segment of the major axis within (bounded by) the ellipse. We now use the fact that $(0,b)$ is on the ellipse, along with the fact that $d=2a$, to get

$$\begin{aligned}\left[\text{distance from } (-c,0) \text{ to } (0,b)\right]+\left[\text{distance from } (c,0) \text{ to } (0,b)\right]&=2a\\ \sqrt{\left(0-(-c)\right)^2+(b-0)^2}+\sqrt{(0-c)^2+(b-0)^2}&=2a \quad \text{distance formula}\\ \sqrt{b^2+c^2}+\sqrt{b^2+c^2}&=2a\\ 2\sqrt{b^2+c^2}&=2a\\ \sqrt{b^2+c^2}&=a\end{aligned}$$

From this, we get $a^2=b^2+c^2$, or $b^2=a^2-c^2$, which will prove useful later on. This implies that $a>b$ and therefore the segment of the major axis within (bounded by) the ellipse is longer than the segment of the minor axis within (bounded by) the ellipse. Now consider a point (x,y) on the ellipse. Applying **Definition 5.4**, we get

$$\left[\text{distance from } (-c,0) \text{ to } (x,y)\right]+\left[\text{distance from } (c,0) \text{ to } (x,y)\right]=2a$$

$$\sqrt{\left(x-(-c)\right)^2+\left(y-0\right)^2}+\sqrt{\left(x-c\right)^2+\left(y-0\right)^2}=2a \quad \text{distance formula}$$

$$\sqrt{\left(x+c\right)^2+y^2}+\sqrt{\left(x-c\right)^2+y^2}=2a$$

In order to make sense of this situation, we use algebra.

$$\sqrt{\left(x+c\right)^2+y^2}+\sqrt{\left(x-c\right)^2+y^2}=2a$$

$$\sqrt{\left(x+c\right)^2+y^2}=2a-\sqrt{\left(x-c\right)^2+y^2}$$

$$\left(\sqrt{\left(x+c\right)^2+y^2}\right)^2=\left(2a-\sqrt{\left(x-c\right)^2+y^2}\right)^2 \quad \text{square both sides}$$

$$\left(x+c\right)^2+y^2=4a^2-4a\sqrt{\left(x-c\right)^2+y^2}+\left(x-c\right)^2+y^2 \quad \text{expand}$$

$$4a\sqrt{\left(x-c\right)^2+y^2}=4a^2+\left(x-c\right)^2-\left(x+c\right)^2 \quad \text{simplify}$$

$$4a\sqrt{\left(x-c\right)^2+y^2}=4a^2-4cx \quad \text{expand and simplify}$$

$$a\sqrt{\left(x-c\right)^2+y^2}=a^2-cx \quad \text{simplify}$$

$$\left(a\sqrt{\left(x-c\right)^2+y^2}\right)^2=\left(a^2-cx\right)^2 \quad \text{square both sides}$$

$$a^2\left(\left(x-c\right)^2+y^2\right)=a^4-2a^2cx+c^2x^2 \quad \text{expand}$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2 \quad \text{expand and distribute}$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2 \quad \text{simplify}$$

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right) \quad \text{factor}$$

We are nearly finished. Recall that $b^2=a^2-c^2$ so that

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$

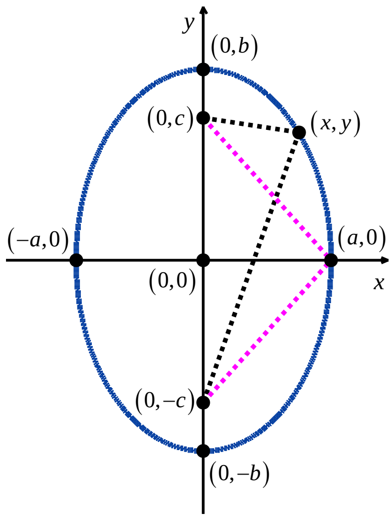
$$b^2x^2+a^2y^2=a^2b^2$$

$$\frac{b^2x^2}{a^2b^2}+\frac{a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We have the formula for an ellipse with its major axis along the *x*-axis. To verify that this formula, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, also applies to an ellipse with its major axis along the *y*-axis, foci at $(0,c)$ and $(0,-c)$, vertices $(0,b)$ and $(0,-b)$, and *x*-intercepts of the ellipse labeled $(a,0)$ and $(-a,0)$, try deriving the formula on your own, using the preceding steps and the following figure as a guide. You will find that, in this case, $b^2=a^2+c^2$.

Figure 5.4. 4



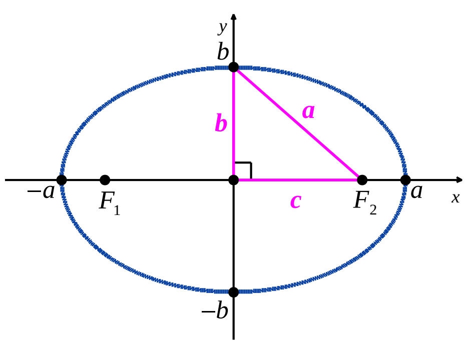
Following is the standard equation for any ellipse with center $(0,0)$.

> **Equation 5.5. The Standard Equation of an Ellipse with Center (0,0):** For positive numbers a and b, $a \neq b$, the equation of an ellipse with center $(0,0)$ is
>
> $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Some remarks about **Equation 5.5** are in order.

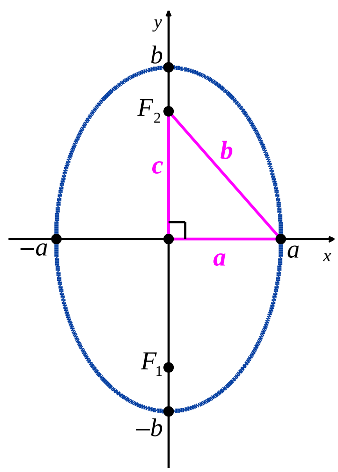
- The values a and b determine how far in the *x*- and *y*-directions, respectively, one counts from the center to arrive at points on the ellipse.

Figure 5.4. 5



$a > b$

Figure 5.4. 6



$b > a$

- If $a > b$, then we have an ellipse whose major axis is horizontal; the foci lie to the left and right of the center. In this case, $a^2 = b^2 + c^2$, as we have seen in the derivation, where c is the distance from the center to a focus. If a and b are known, then c may be found by $c = \sqrt{a^2 - b^2}$.
- If $b > a$, the roles of the major and minor axes are reversed and the foci lie above and below the center. In this case, $b^2 = a^2 + c^2$, from which $c = \sqrt{b^2 - a^2}$.
- In either case, c is the distance from the center to each focus, and for numbers a and b, $c = \sqrt{(\text{larger number})^2 - (\text{smaller number})^2}$

Example 5.4.1. Identify the center, the vertices, and the foci of the ellipse given by the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Graph the ellipse.

Solution. The equation is in the standard form given by **Equation 5.5**. Thus, the ellipse has its center at $(0,0)$. We see that $a^2 = 16$ and $b^2 = 9$, from which $a = 4$ and $b = 3$. We move 4 units left and right from the center and 3 units up and down from the center to arrive at points on the ellipse, as seen below on the left.

Figure 5.4. 7

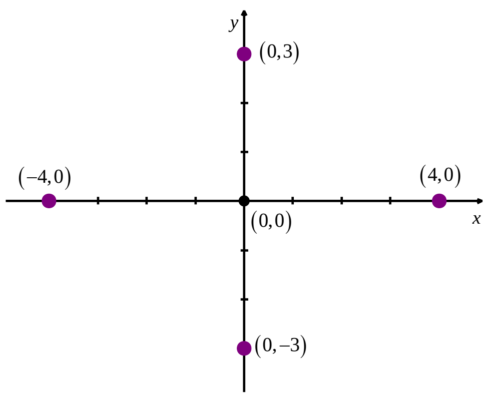
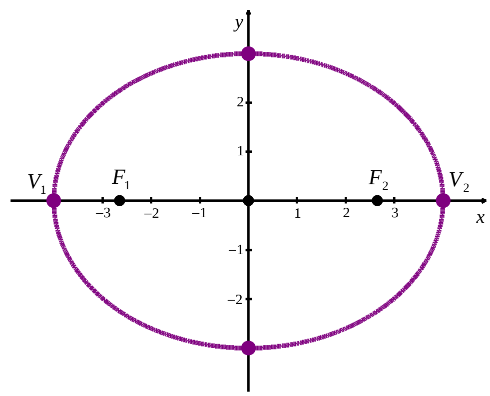


Figure 5.4. 8



Since we moved farther in the *x*-direction than in the *y*-direction, the major axis is the *x*-axis and the minor axis is the *y*-axis. The vertices are the points of intersection of the ellipse with the major axis, so in this case they are the points $(-4,0)$ and $(4,0)$. To find the foci, we first evaluate *c* using $c = \sqrt{a^2 - b^2}$:

$$c = \sqrt{16 - 9} = \sqrt{7}$$

Thus, the foci lie $\sqrt{7}$ units left and right of the center, on the major axis, at $(-\sqrt{7},0)$ and $(\sqrt{7},0)$.

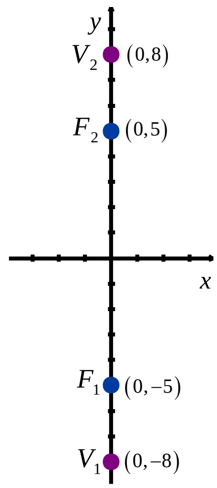
Plotting all of this information, and connecting the four points, $(4,0)$, $(0,3)$, $(-4,0)$, and $(0,-3)$, with a smooth curve to form the ellipse gives the graph seen above on the right.

□

Example 5.4.2. Find the standard equation of the ellipse that has vertices $(0,\pm 8)$ and foci $(0,\pm 5)$.

Solution. Plotting the data given to us, we have the scenario shown below.

Figure 5.4. 9



From this sketch, we deduce that the major axis is the *y*-axis. Since the center is the midpoint of the foci, we find it is $(0,0)$. Thus, we can use **Equation 5.5**, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The vertex $(0,8)$, being 8 units from the center, gives us $b=8$, and so $b^2=64$. All that remains is to find a^2. From the focus at $(0,5)$, it follows that $c=5$. Noting that the major axis is the *y*-axis, and thus $b>a$, we have

$$\begin{aligned} c &= \sqrt{b^2-a^2} \\ 5 &= \sqrt{64-a^2} \\ 25 &= 64-a^2 \qquad \text{after squaring both sides} \\ a^2 &= 39 \end{aligned}$$

Substituting $a^2=39$ and $b^2=64$ into the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation is $\frac{x^2}{39}+\frac{y^2}{64}=1$.

□

The Equation of an Ellipse with Center (h,k)

To get the formula for the ellipse centered at (h,k), we could use transformations, or re-derive the equation using **Definition 5.4** and the distance formula.

Equation 5.6. The Standard Equation of an Ellipse with Center (h,k): For positive numbers a and b, $a \neq b$, the equation of an ellipse with center (h,k) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

We note that a center of $(h,k)=(0,0)$ results in **Equation 5.5**.

Example 5.4.3. Identify the center, the major and minor axes, the vertices, the points of the ellipse that lie on the minor axis, and the foci of the ellipse given by the equation $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{25}=1$. Graph the ellipse.

Solution. We see that this formula is in the standard form of **Equation 5.6**. Here $x-h$ is $x+1$, so $h=-1$, and $y-k$ is $y-2$, so $k=2$; the ellipse is centered at $(-1,2)$. From $a^2=9$ and $b^2=25$, we get $a=3$ and $b=5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse, as shown in the following diagram, on the left.

Figure 5.4. 10

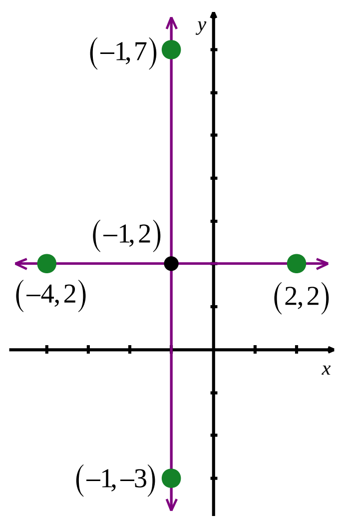
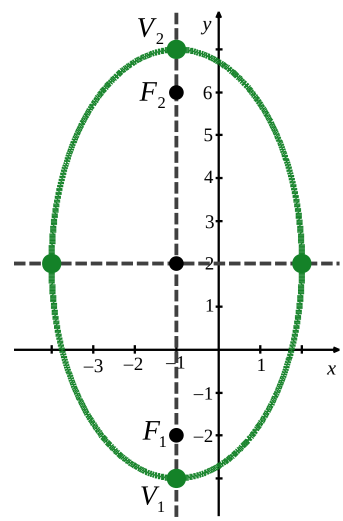


Figure 5.4. 11



The major and minor axes cross at the center, $(-1,2)$. With $b>a$, since we moved farther from the center in the *y*-direction than in the *x*-direction, the major axis is the vertical line $x=-1$, and the minor axis is the horizontal line $y=2$. The vertices are the points on the ellipse that intersect the major axis so, in this case, they are the points $(-1,7)$ and $(-1,-3)$. The points of the ellipse that lie on the minor axis are $(-4,2)$ and $(2,2)$. To locate the foci, we find

$$\begin{aligned} c &= \sqrt{b^2 - a^2} \\ &= \sqrt{25 - 9} \\ &= 4 \end{aligned}$$

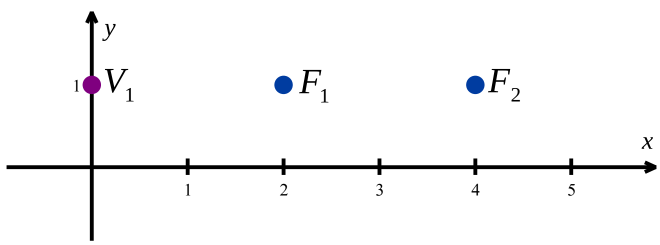
Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1,6)$ and $(-1,-2)$. Plotting all this information and connecting the points $(2,2)$, $(-1,7)$, $(-4,2)$, and $(-1,-3)$ with a smooth curve to form the ellipse gives the graph seen to the right of the previous diagram.

□

Example 5.4.4. Find the equation of the ellipse with foci $(2,1)$ and $(4,1)$, and vertex $(0,1)$.

Solution. Plotting the data given to us, we have the following.

Figure 5.4. 12



From this sketch, we know that the major axis is horizontal, meaning $a > b$. Since the center is the midpoint of the foci, it is the point $(3,1)$. The one vertex we are given, $(0,1)$, is three units from the center, so $a = 3$, and $a^2 = 9$. All that remains is to find b^2. The foci are one unit away from the center, and so we know $c = 1$. Since $a > b$, we have

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ 1 &= \sqrt{9 - b^2} \\ 1 &= 9 - b^2 \qquad \text{after squaring both sides} \\ b^2 &= 8 \end{aligned}$$

Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get a final answer of

$$\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1.$$

□

As with circles and parabolas, an equation may be given that represents an ellipse, but that is not in the standard form of **Equation 5.6**. In those cases, as with circles and parabolas before, we will need to transform the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

1. Position all terms containing variables on the left side of the equation, grouping terms with 'like' variables together. Position the constant, if any, on the right side.
2. Complete the square in both variables as needed.
3. Divide both sides of the equation by the constant on the right side,[10] resulting in 1 on the right side.[11]

Example 5.4.5. Put the equation $x^2+4y^2-2x+24y+33=0$ into standard form. Find the center, the vertices, and the foci, and use these in graphing the ellipse.

Solution. We position the constant on the right side, complete both squares, and then divide to get the right side equal to 1.

$$\begin{aligned}
x^2+4y^2-2x+24y+33&=0 \\
x^2-2x+4y^2+24y&=-33 && \text{group variables and subtract 33 from both sides} \\
x^2-2x+4\left(y^2+6y\right)&=-33 && \text{factor out leading coefficient from } y\text{-terms} \\
\left(x^2-2x+1\right)+4\left(y^2+6y+9\right)&=-33+1+4(9) && \text{complete the square in both variables} \\
(x-1)^2+4(y+3)^2&=4 && \text{factor and simplify} \\
\frac{(x-1)^2}{4}+\frac{4(y+3)^2}{4}&=\frac{4}{4} && \text{divide both sides by 4} \\
\frac{(x-1)^2}{4}+\frac{(y+3)^2}{1}&=1 && \text{simplify}
\end{aligned}$$

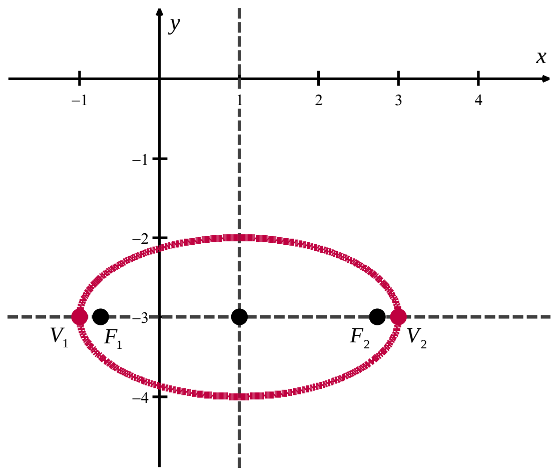
Now that this equation is in the standard form of **Equation 5.6**, we see that $x-h$ is $x-1$, so $h=1$, and $y-k$ is $y+3$, so $k=-3$. Hence, our ellipse is centered at $(1,-3)$. We see that $a^2=4$ and $b^2=1$, from which $a=2$ and $b=1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the *x*-direction than in the *y*-direction, the major axis is the horizontal line $y=-3$.

The vertices are the points on the ellipse that lie along the major axis so they are the points $(-1,-3)$ and $(3,-3)$. To determine the foci, we find $c=\sqrt{4-1}=\sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the

[10] If the constant on the right side is zero, either a mistake has been made or the equation does not represent a proper ellipse. In this case, the equation represents a single point.
[11] If, after Step 3, the constant term is 1 but the left side of the equation now contains two negative square terms, the equation does not represent a proper ellipse.

center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $\left(1-\sqrt{3},-3\right)$ and $\left(1+\sqrt{3},-3\right)$. Plotting all of this information gives the following:

Figure 5.4. 13



□

The Eccentricity of an Ellipse

As you work more with ellipses, you will notice they come in different shapes and sizes. Compare the two ellipses below.

Figure 5.4. 14

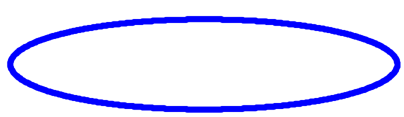
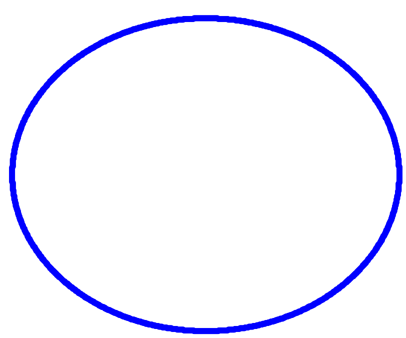


Figure 5.4. 15



Certainly, one ellipse is more round than the other. This notion of roundness is quantified below.

Definition 5.5. The **eccentricity** of an ellipse, denoted *e*, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

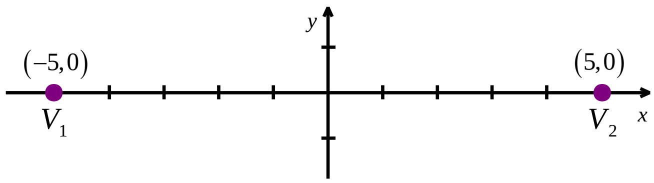
In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.97$; for the ellipse on the right $e \approx 0.58$. In general, the closer the eccentricity is to

0, the more ‘circular’ the ellipse will appear; the closer the eccentricity is to 1, the more ‘flat’ the ellipse will appear.

Example 5.4.6. Find the equation of the ellipse whose vertices are $(\pm 5,0)$, with eccentricity $e=\frac{1}{4}$.

Solution. As before, we plot the data given to us.

Figure 5.4. 16



From this sketch, we deduce that the major axis is horizontal, meaning $a>b$. With the vertices located at $(-5,0)$ and $(5,0)$, we find the center is $(0,0)$, since the center is the midpoint of the vertices. We also get $a=5$, so $a^2=25$. To find b^2, we use the eccentricity, $e=\frac{1}{4}$, to find c.

$$e=\frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}=\frac{c}{a}$$
$$\frac{1}{4}=\frac{c}{5}$$

Then $c=\frac{5}{4}$ and we proceed to find b^2.

$$c=\sqrt{a^2-b^2}$$
$$\frac{5}{4}=\sqrt{25-b^2}$$
$$\frac{25}{16}=25-b^2 \qquad \text{after squaring both sides}$$

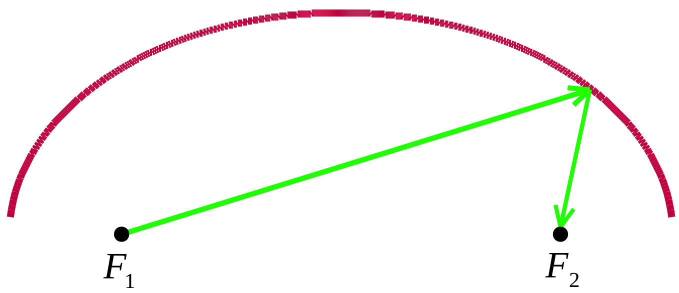
It follows that $b^2=25-\frac{25}{16}=\frac{375}{16}$. Substituting all of our findings into **Equation 5.5**, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the final answer is $\frac{x^2}{25}+\frac{16y^2}{375}=1$.

□

Applications of Ellipses

As with parabolas, ellipses have a reflective property. If we imagine the green lines below representing sound waves, then the waves emanating from one focus reflect off the ellipse and head toward the other focus.

Figure 5.4. 17

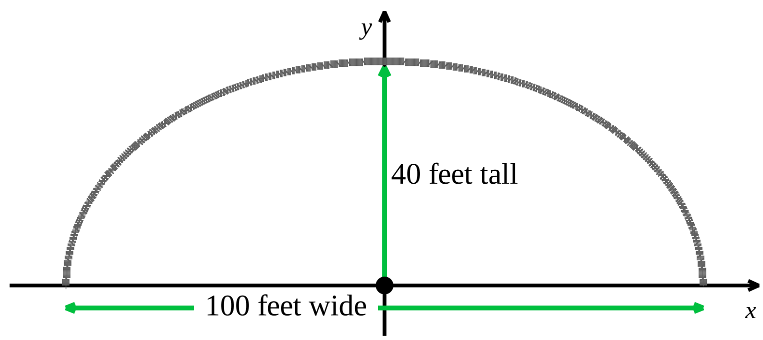


Such geometry is exploited in the construction of so-called 'whispering galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore whispering galleries in our last example.

Example 5.4.7. Shawna and Spencer want to exchange secrets from across a crowded whispering gallery, which is a room having a cross section that is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Graphing the data yields the following.

Figure 5.4. 18



It is most convenient to imagine this ellipse centered at $(0,0)$. Since the ellipse is 100 feet wide and 40 feet tall, we get $a=50$ and $b=40$. Hence, our ellipse has the equation $\dfrac{x^2}{50^2}+\dfrac{y^2}{40^2}=1$. We are looking for the foci and we begin by finding $c=\sqrt{a^2-b^2}$.

$$\begin{aligned} c &= \sqrt{50^2-40^2} \\ &= \sqrt{900} \\ &= 30 \end{aligned}$$

The foci are 30 feet from the center. That means they are $50-30=20$ feet from the vertices. So Shawna and Spencer should stand on opposite ends of the gallery (along the major axis), each 20 feet from the outside wall.

□

5.4 Exercises

1. Define an ellipse in terms of its foci.
2. What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the *y*-axis?

In Exercises 3 – 8, write the equation in standard form if it represents an ellipse. If the equation does not represent an ellipse, explain how the equation violates the definition of an ellipse.

3. $2x^2 + y = 4$
4. $4x^2 + 9y^2 = 36$
5. $4x^2 - y^2 = 4$
6. $4x^2 + 9y^2 = 1$
7. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$
8. $4x^2 + 4y^2 = 1$

In Exercises 9 – 20, find the center, the vertices, and the foci of the ellipse. Graph the ellipse.

9. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
10. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
11. $\dfrac{x^2}{25} + \dfrac{y^2}{36} = 1$
12. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$
13. $\dfrac{(x-2)^2}{64} + \dfrac{(y-4)^2}{16} = 1$
14. $\dfrac{x^2}{2} + \dfrac{(y+1)^2}{5} = 1$
15. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
16. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
17. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
18. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
19. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
20. $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

In Exercises 21 – 30, put the equation of the ellipse in standard form. Find the center, the vertices, and the foci. Graph the ellipse.

21. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
22. $12x^2 + 3y^2 - 30y + 39 = 0$
23. $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
24. $x^2 - 2x + 2y^2 - 12y + 3 = 0$
25. $9x^2 + 4y^2 - 4y - 8 = 0$
26. $6x^2 + 5y^2 - 24x + 20y + 14 = 0$
27. $4x^2 - 24x + 36y^2 - 360y + 864 = 0$
28. $4x^2 + 24x + 16y^2 - 128y + 228 = 0$
29. $4x^2 + 40x + 25y^2 - 100y + 100 = 0$
30. $9x^2 + 72x + 16y^2 + 16y + 4 = 0$

In Exercises 31 – 42, find the standard form of the equation of the ellipse that has the given properties.

31. Center $(3,7)$, Vertex $(3,2)$, Focus $(3,3)$

32. Center $(4,2)$, Vertex $(9,2)$, Focus $(4+2\sqrt{6},2)$

33. Center $(3,5)$, Vertex $(3,11)$, Focus $(3,5+4\sqrt{2})$

34. Center $(-3,4)$, Vertex $(1,4)$, Focus $(-3+2\sqrt{3},4)$

35. Foci $(0,\pm 5)$, Vertices $(0,\pm 8)$

36. Foci $(\pm 3,0)$; Segment of the minor axis within (bounded by) the ellipse has length 10.

37. Vertices $(3,2)$ and $(13,2)$; Minor axis intersects ellipse at points $(8,4)$ and $(8,0)$

38. Center $(5,2)$, Vertex $(0,2)$, eccentricity $\frac{1}{2}$

39. All points on the ellipse are in Quadrant IV except $(0,-9)$ and $(8,0)$. (One might also say that the ellipse is tangent to the axes at those two points.)

40. Center $(0,0)$, Focus $(4,0)$, containing the point $(0,3)$

41. Center $(0,0)$, Focus $(0,-2)$, containing the point $(5,0)$

42. Center $(0,0)$, Focus $(3,0)$; The segment of the major axis within (bounded by) the ellipse is twice as long as the segment of the minor axis within (bounded by) the ellipse.

In Exercises 43 – 47, given the graph of the ellipse, determine its equation.

43.

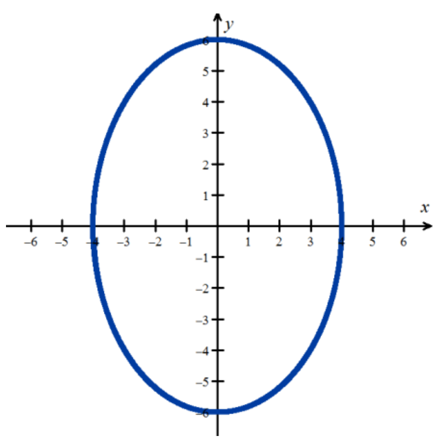


Figure Ex5.4.1

44.

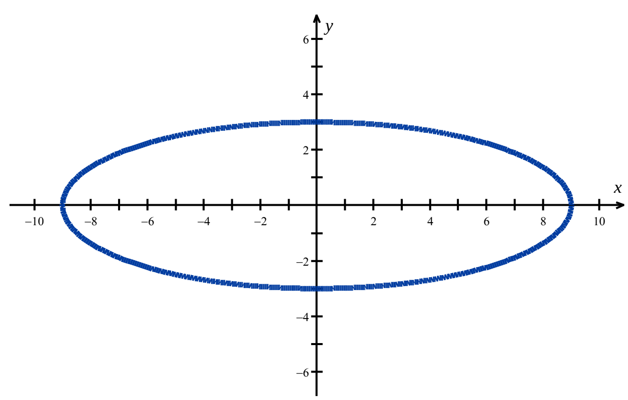


Figure Ex5.4.2

45.

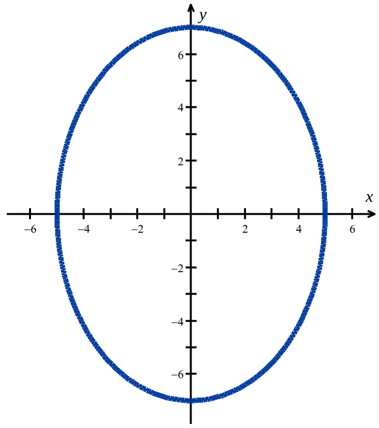


Figure Ex5.4. 3

46.

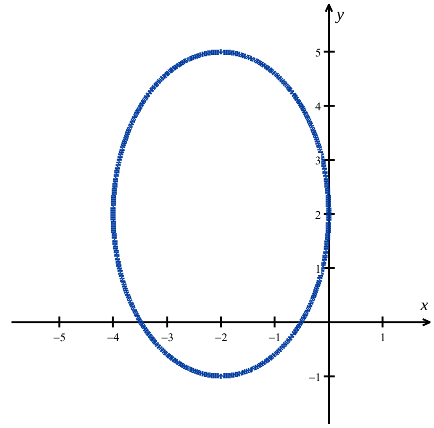


Figure Ex5.4.4

47.

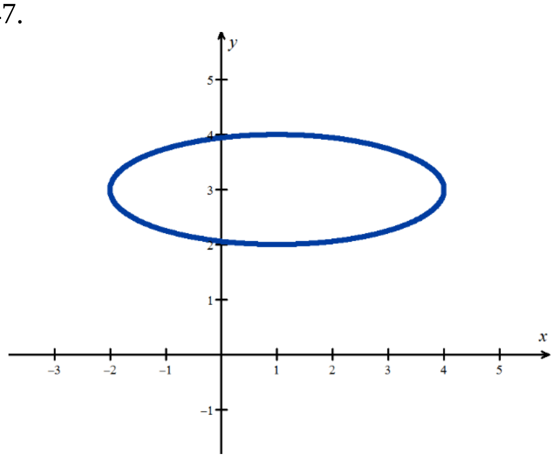


Figure Ex5.4.5

48. Find the equation of the ellipse whose vertices are $(\pm 12, 0)$ with eccentricity $e = \frac{1}{6}$.

49. Find the equation of the ellipse whose vertices are $(0, \pm 4)$ with eccentricity $e = \frac{1}{3}$.

50. An elliptical arch[12] has a height of 8 feet and a span of 20 feet. Placing the arch on a coordinate plane with highest point at $(0,8)$, find an equation for the ellipse and use it to find the height of the arch at a distance of 4 feet from the center.

51. An elliptical arch has a height of 12 feet and a span of 40 feet. Placing the arch on a coordinate plane with highest point at $(0,12)$, find an equation for the ellipse and use it to find the distance from the center to a point at which the height is 6 feet.

[12] An elliptical arch is an arch in the shape of a semi-ellipse, or the top half of an ellipse.

52. A bridge is to be built in the shape of an elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.
53. An elliptical arch is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 39 in **Section 5.3**.
54. A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.
55. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length of the gallery and what is its height at the center?
56. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.
57. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, and National Statuary Hall in Washington D.C. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in **Example 5.4.7**?
58. The points $(1,-4)$, $(3,-1)$, $(5,-4)$, $(3,-7)$, $\left(\frac{-2\sqrt{5}+9}{3},-2\right)$, and $\left(\frac{2\sqrt{5}+9}{3},-2\right)$ lie on the ellipse $\frac{(x-3)^2}{4}+\frac{(y+4)^2}{9}=1$. Find three other points that lie on the ellipse.

5.5 Hyperbolas

Learning Objectives

- Define a hyperbola in the plane.
- Write the equation of a hyperbola in standard form.
- Graph a hyperbola from a given equation.
- Determine the center, vertices, and foci of a hyperbola.
- Find the equation of a hyperbola from a graph or from stated properties.
- Solve application problems modeled by hyperbolas.
- Determine the type of conic section that an equation represents.

As stated in **Definition 5.4**, an ellipse is the collection of points whose sum of distances from two distinct points is the same. You may wonder what, if any, curve we would have if we replaced **sum** with **difference**. The answer is a hyperbola.

The Definition of a Hyperbola

Definition 5.6. Given two distinct points F_1 and F_2 in the plane and a positive number d that is smaller than the distance between F_1 and F_2, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of the distance from F_1 to (x, y) and the distance from F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.

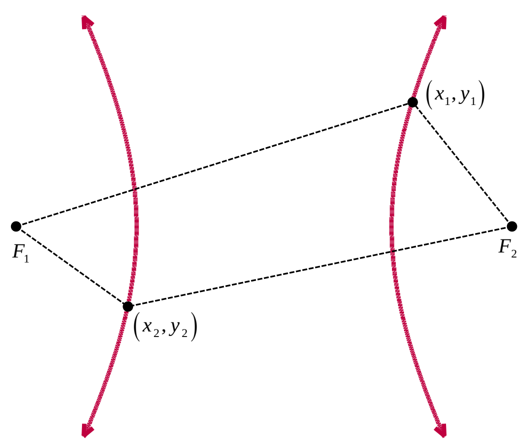


Figure 5.5. 1

In the figure,

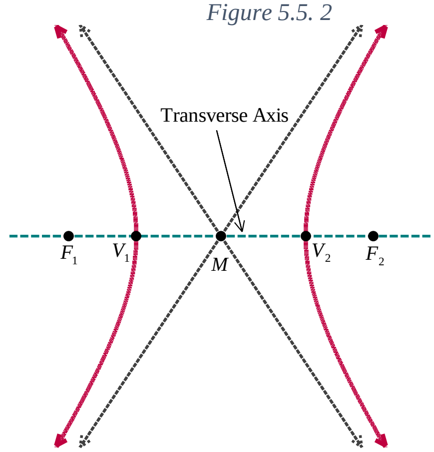
$$\left[\text{the distance from } F_1 \text{ to } (x_1, y_1)\right] - \left[\text{the distance from } F_2 \text{ to } (x_1, y_1)\right] = d$$

and

$$\left[\text{the distance from } F_2 \text{ to } (x_2, y_2)\right] - \left[\text{the distance from } F_1 \text{ to } (x_2, y_2)\right] = d$$

A hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment joining the two foci. The **transverse axis** of the hyperbola is the line through the center and foci of the hyperbola. The **vertices** of a hyperbola are the points of intersection of the hyperbola with the transverse axis. In addition, hyperbolas have two asymptotes. These serve as guides to the graph.

Figure 5.5. 2



A hyperbola with center M, foci F_1 and F_2, vertices V_1 and V_2

Note that the center is also the midpoint of the line segment joining the two vertices, as shown below.

$$\begin{aligned}
V_1F_2 - V_1F_1 &= V_2F_1 - V_2F_2 && \text{definition of a hyperbola}\\
V_1V_2 + V_2F_2 - V_1F_1 &= V_2V_1 + V_1F_1 - V_2F_2 &&\\
V_2F_2 - V_1F_1 &= V_1F_1 - V_2F_2 &&\\
V_1F_1 &= V_2F_2 &&\\
MF_1 - MV_1 &= MF_2 - MV_2 &&\\
MV_1 &= MV_2 && M \text{ is center of foci so } MF_1 = MF_2
\end{aligned}$$

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line through the center that is perpendicular to the transverse axis.

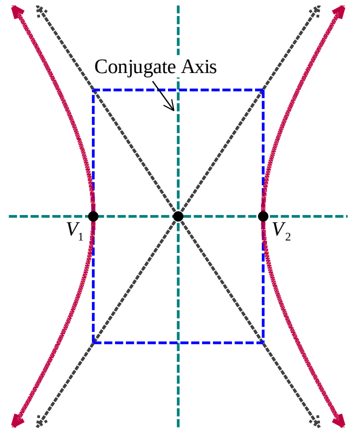


Figure 5.5. 3

Note the rectangle that is sketched in the above diagram. Two of the line segments that compose this rectangle are parallel to the conjugate axis, pass through the vertices, and have endpoints on the asymptotes. The other two line segments, which share these endpoints, are parallel to the transverse axis. This **guide rectangle** will aid us in graphing hyperbolas.

The Equation of a Hyperbola with Center at $(0,0)$

To derive the equation of a hyperbola, for simplicity, we assume that the center is $(0,0)$, the vertices are $(a,0)$ and $(-a,0)$, and the foci are $(c,0)$ and $(-c,0)$. We label the points where the conjugate axis intersects the guide rectangle as $(0,b)$ and $(0,-b)$. (This is not the definition of b. We will demonstrate how these points of intersection are found later.) Schematically, we have the following.

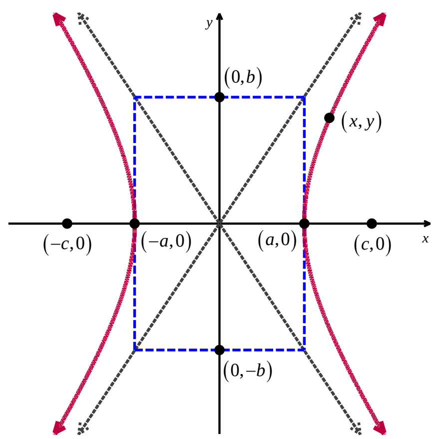


Figure 5.5. 4

Since $(a,0)$ is on the hyperbola, it must satisfy the conditions of **Definition 5.6**. That is, the distance from $(-c,0)$ to $(a,0)$ minus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all these points lie on the *x*-axis, we have

$$\begin{aligned}\left[\text{distance from } (-c,0) \text{ to } (a,0)\right]-\left[\text{distance from } (c,0) \text{ to } (a,0)\right]&=d\\(a+c)-(c-a)&=d\\2a&=d\end{aligned}$$

In other words, the positive number d from the definition of the hyperbola is actually the length of the line segement connecting the two vertices. (Where have we seen that type of coincidence before?) Now consider a point (x,y) on the hyperbola. Applying **Definition 5.6**, we get

$$\begin{aligned}\left[\text{distance from } (-c,0) \text{ to } (x,y)\right]-\left[\text{distance from } (c,0) \text{ to } (x,y)\right]&=2a \quad \text{use } d=2a\\\sqrt{(x-(-c))^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}&=2a \quad \text{apply distance formula}\\\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}&=2a\end{aligned}$$

Using algebra, as we did in deriving the standard formula of an ellipse, **Equation 5.5**, we arrive at the following.[13]

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$

We can simplify this equation further. To that end, we note that since a and c are both positive numbers with $a<c$, we get $a^2<c^2$ so that a^2-c^2 is a negative number. Hence, c^2-a^2 is a positive number. Let $b=\sqrt{c^2-a^2}$, or $b^2=c^2-a^2$ with $b>0$.

$$\begin{aligned}-\left(c^2-a^2\right)x^2+a^2y^2&=-a^2\left(c^2-a^2\right) \quad \text{factor to obtain positive values}\\-b^2x^2+a^2y^2&=-a^2b^2\\\frac{-b^2x^2}{-a^2b^2}+\frac{a^2y^2}{-a^2b^2}&=\frac{-a^2b^2}{-a^2b^2}\\\frac{x^2}{a^2}-\frac{y^2}{b^2}&=1\end{aligned}$$

Thus, the equation for the hyperbola, as defined above, is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. To show that $y=\pm\frac{b}{a}x$ are the asymptotes, we must show that $\left[y-\left(\pm\frac{b}{a}x\right)\right]\to 0$ as $x\to\pm\infty$. An equivalent proof is to show that $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$ as $x\to\pm\infty$, and this proof follows.

[13] It is a good exercise to actually work this out.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
$$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$$
$$\frac{y^2}{x^2}=\frac{b^2}{a^2}-\frac{b^2}{x^2} \text{ multiply both sides by } \frac{b^2}{x^2}$$

Now, $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$ as $x\to\pm\infty$ since $\frac{b^2}{x^2}\to 0$ as $x\to\pm\infty$. Notice that the points (a,b) and $(a,-b)$ are on the asymptotes $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$, respectively. This justifies our earlier usage of $(0,\pm b)$ as the points where the conjugate axis intersects the guide rectangle.

The equation above is for a hyperbola whose center is the origin, and which opens to the left and right. We refer to a hyperbola that opens to the left and right as a **horizontal hyperbola.**

Equation 5.7. The Standard Equation of a Horizontal Hyperbola with Center (0,0): For positive numbers a and b, the equation of a horizontal hyperbola with center $(0,0)$ is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

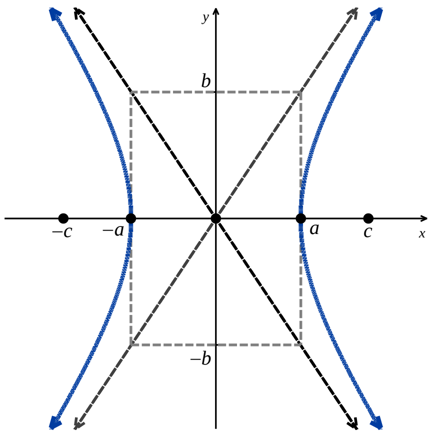
If the roles of x and y are interchanged, the hyperbola's branches open upward and downward, resulting in a **vertical hyperbola**.

Equation 5.8. The Standard Equation of a Vertical Hyperbola with Center (0,0): For positive numbers a and b, the equation of a vertical hyperbola with center $(0,0)$ is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

For both horizontal and vertical hyperbolas, the values of a and b determine how far in the *x*- and *y*-directions, respectively, one counts from the center to determine the rectangle whose diagonals lie on the asymptotes. In both cases, the distance c from the center to the foci, as seen in the derivation, can be found by the formula $c=\sqrt{a^2+b^2}$.

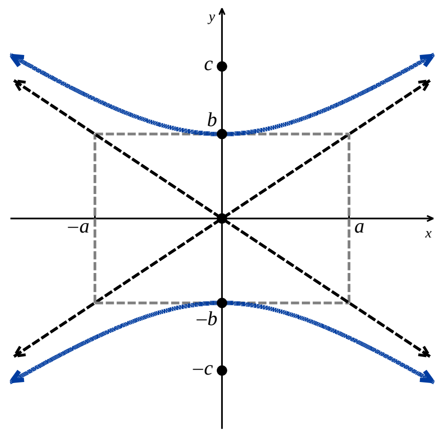
Figure 5.5. 5



Horizontal hyperbola with center $(0,0)$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Figure 5.5. 6



Vertical hyperbola with center $(0,0)$

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a **difference** of squares while the circle and ellipse formulas both involve the **sum** of squares.

Example 5.5.1. Identify the vertices and the foci of the hyperbola given by the equation $\frac{y^2}{49}-\frac{x^2}{32}=1$.

Graph the hyperbola.

Solution. We first see that this equation is given to us in the standard form of **Equation 5.8**. Hence, this is a vertical hyperbola centered at $(0,0)$. We see that $a^2=32$, so $a=4\sqrt{2}$, and $b^2=49$, so $b=7$. This means we move $4\sqrt{2}$ units left and right of the center and 7 units up and down from the center to arrive at points on a guide rectangle. The asymptotes contain the diagonals of the rectangle. This results in the following set up.

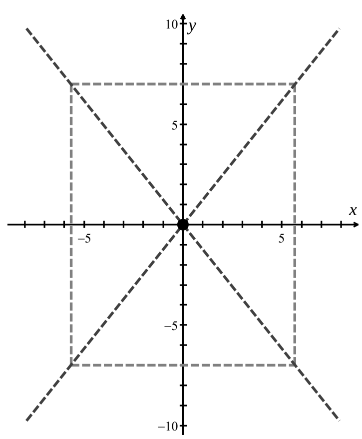


Figure 5.5. 7

We note that the x^2 term is being subtracted from the y^2 term, the branches of the hyperbola open upward and downward, and the transverse axis is the *y*-axis. The vertices of the hyperbola are the points where the hyperbola intersects the transverse axis. In this case, the vertices are $(0,\pm 7)$. To find the foci, we first determine c.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{32+49} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

The foci lie on the transverse axis, so we move 9 units upward and downward from the center $(0,0)$ to arrive at foci of $(0,\pm 9)$. The graph of the hyperbola passes through vertices and approaches asymptotes.

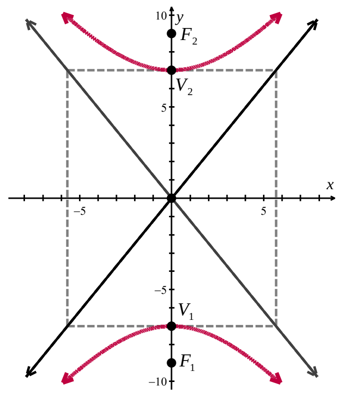


Figure 5.5. 8

□

Example 5.5.2. Find the standard form of the equation of the hyperbola that has vertices $(\pm 6,0)$ and foci $(\pm 2\sqrt{10},0)$.

Solution. Since the vertices and foci are on the *x*-axis, and the center is midway between the foci at $(0,0)$, the equation for the hyperbola will have the form of **Equation 5.7**, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. From the vertices, $(\pm 6,0)$, we see that $a=6$ so $a^2=36$. The foci, $(\pm 2\sqrt{10},0)$, give us $c=2\sqrt{10}$ so we have $c^2=40$. We next find b^2.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ c^2 &= a^2+b^2 && \text{square both sides} \\ b^2 &= c^2-a^2 \\ &= 40-36 \\ &= 4 \end{aligned}$$

Finally, substituting $a^2 = 36$ and $b^2 = 4$ into **Equation 5.7** yields $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

□

The hyperbola from **Example 5.5.2** is graphed below. The guide rectangle, determined by the values $a = 6$ and $b = 2$, is included, along with the asymptotes.

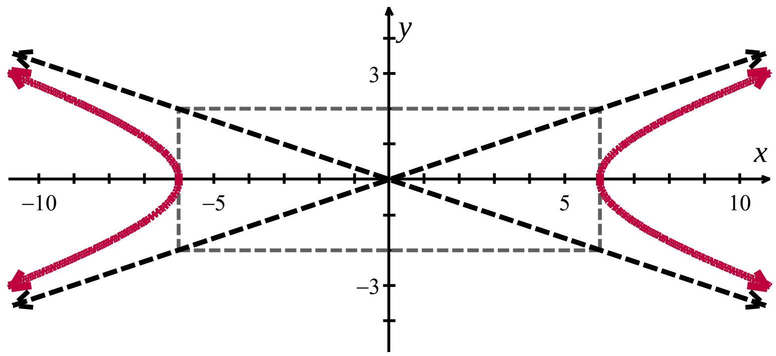


Figure 5.5. 9

The Equation of a Hyperbola with Center at (h, k)

Following is an equation for a horizontal hyperbola, whose branches open to the left and right, and which is centered at the point (h, k).

Equation 5.9. The Standard Equation of a Horizontal Hyperbola with Center (h , k): For positive numbers a and b, the equation of a horizontal hyperbola with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Exchanging the roles of x and y results in a vertical hyperbola, whose branches open upward and downward.

Equation 5.10. The Standard Equation of a Vertical Hyperbola with Center (h , k): For positive numbers a and b, the equation of a vertical hyperbola with center (h, k) is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

We note that a center of $(h, k) = (0, 0)$ in these two equations results in **Equations 5.7** and **5.8**. Also, for both horizontal and vertical hyperbolas with center (h, k), the distance c from the center to the foci can be found by the formula $c = \sqrt{a^2 + b^2}$.

Example 5.5.3. Find the center, the vertices, the foci, and the equations of the asymptotes for the hyperbola given by the equation $\frac{(x-2)^2}{4}-\frac{y^2}{25}=1$. Graph the hyperbola.

Solution. We first see that this equation is given to us in the standard form of **Equation 5.9**. Here $x-h$ is $x-2$, so $h=2$, and $y-k$ is y, so $k=0$. Hence, our hyperbola is centered at $(2,0)$. We see that $a^2=4$, so $a=2$, and $b^2=25$, so $b=5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes contain the diagonals of the rectangle, as displayed below.

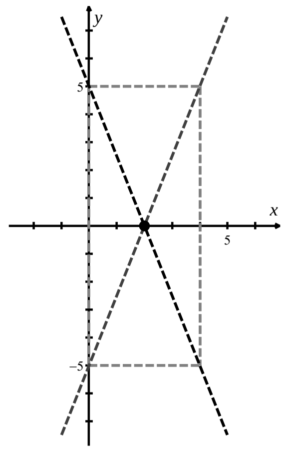


Figure 5.5. 10

Since the y^2 term is being subtracted from the x^2 term, the branches of the hyperbola open to the left and right. This means that the transverse axis is the *x*-axis. Knowing that the vertices of the hyperbola are the points where the hyperbola intersects the transverse axis, we plot the vertices 2 units to the left and 2 units to the right of the center $(2,0)$, at the points $(0,0)$ and $(4,0)$.

To find the foci, we determine *c*.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{4+25} \\ &= \sqrt{29} \end{aligned}$$

Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of the center $(2,0)$ to arrive at the points $(2-\sqrt{29},0)$ and $(2+\sqrt{29},0)$, approximately $(-3.39,0)$ and $(7.39,0)$, respectively.

We next determine the equations of the asymptotes, recalling that the asymptotes contain the diagonals of the guide rectangle, so they have slopes of $\pm\frac{b}{a}=\pm\frac{5}{2}$. Using the point-slope equation of a line yields $y-0=\pm\frac{5}{2}(x-2)$, so the asymptotes are $y=\frac{5}{2}x-5$ and $y=-\frac{5}{2}x+5$.

Putting all of this together, we get the following graph.

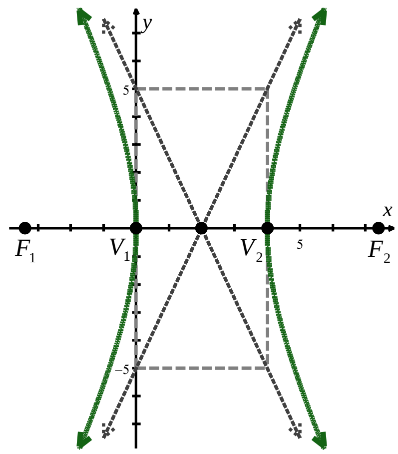


Figure 5.5. 11

□

Example 5.5.4. Find the equation of the hyperbola with asymptotes $y=\pm 2x$ and vertices $(\pm 5,0)$.

Solution. Plotting the data given to us, we have the following.

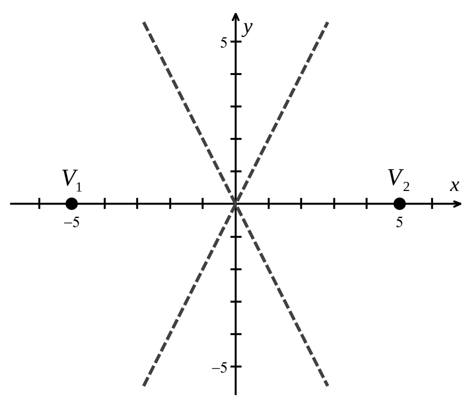


Figure 5.5. 12

This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0,0)$. Hence, the standard form is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. From the vertices of $(\pm 5,0)$, it follows that $a=5$, so $a^2=25$. In order to determine b^2, we recall that the slopes of the asymptotes are

$\pm\dfrac{b}{a}$. Since $a=5$ and the slope of the line $y=2x$ is 2, we have $\dfrac{b}{5}=2$, so $b=10$. Hence, $b^2=100$ and our final answer is $\dfrac{x^2}{25}-\dfrac{y^2}{100}=1$.

□

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Position all terms containing variables on the left side of the equation, grouping terms with 'like' variables together. Position the constant, if any, on the right side.
2. Complete the square in both variables as needed.
3. Divide both sides of the equation by the constant on the right side, resulting in 1 on the right side.[14]

Example 5.5.5. Put the equation $9y^2-x^2-6x=10$ into standard form. Find the center, the vertices, the foci, and the equations of the asymptotes. Graph the hyperbola.

Solution. We need only complete the square for the *x*-terms.

$$9y^2-x^2-6x=10$$
$$9y^2-1\left(x^2+6x\right)=10$$
$$9y^2-1\left(x^2+6x+9\right)=10-1(9) \quad \text{complete the square in } x$$
$$9y^2-(x+3)^2=1 \quad \text{factor and simplify}$$
$$\frac{y^2}{\frac{1}{9}}-\frac{(x+3)^2}{1}=1 \quad \text{write in the standard form for a vertical hyperbola}$$

Now that this equation is in the standard form of **Equation 5.10**, we see that $x-h$ is $x+3$, so $h=-3$, and $y-k$ is y, so $k=0$. Hence, the hyperbola is centered at $(-3,0)$. We find that $a^2=1$, so $a=1$, and $b^2=\dfrac{1}{9}$, so $b=\dfrac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\dfrac{1}{3}$ unit up and down from the center to arrive at points on the guide rectangle.

[14] If the constant on the right side is zero, either a mistake has been made or the equation does not represent a hyperbola. In this case, the equation represents a pair of lines.

Since the x^2 term is being subtracted from the y^2 term, the branches of the hyperbola open upward and downward. This means the transverse axis is the vertical line $x=-3$ and the conjugate axis is the *x*-axis. Since the vertices of the hyperbola are points where the hyperbola intersects the transverse axis, the vertices are $\frac{1}{3}$ of a unit above and below the center $(-3,0)$, at the points $\left(-3,\frac{1}{3}\right)$ and $\left(-3,-\frac{1}{3}\right)$. To find the foci, we first determine c.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{1+\frac{1}{9}} \\ &= \frac{\sqrt{10}}{3} \end{aligned}$$

The foci lie on the transverse axis, so we move $\frac{\sqrt{10}}{3}$ units above and below the center $(-3,0)$ to arrive at foci of $\left(-3,\frac{\sqrt{10}}{3}\right)$ and $\left(-3,-\frac{\sqrt{10}}{3}\right)$.

To determine the asymptotes, recall that the asymptotes contain the diagonals of the guide rectangle, so they have slopes of $\pm\frac{b}{a}=\pm\frac{1}{3}$. We use the point-slope equation of a line to find the asymptotes.

$$\begin{aligned} y-0 &= \frac{1}{3}(x-(-3)) \\ y &= \frac{1}{3}x+1 \end{aligned} \qquad\qquad \begin{aligned} y-0 &= -\frac{1}{3}(x-(-3)) \\ y &= -\frac{1}{3}x-1 \end{aligned}$$

Putting it all together results in the following graph.

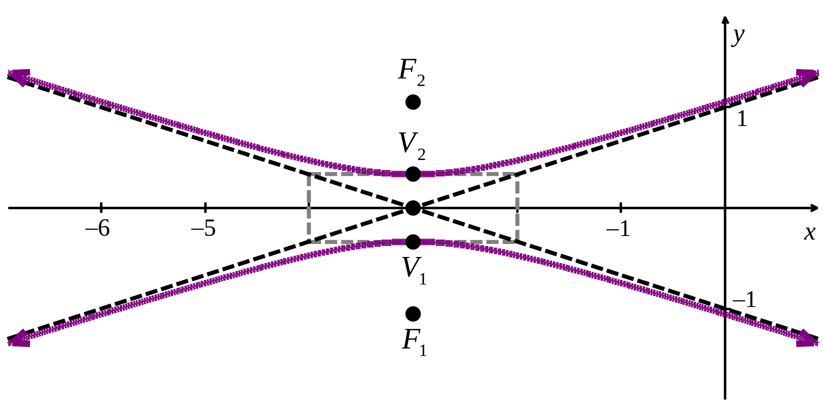


Figure 5.5. 13

□

Applications of Hyperbolas

Hyperbolas have real-world applications in many fields. They can be used to model the paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength.

Figure 5.5. 14

Example 5.5.6. The design layout of a hyperbolic cooling tower is shown below. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola is the origin of the coordinate plane. Round final values to four decimal places.

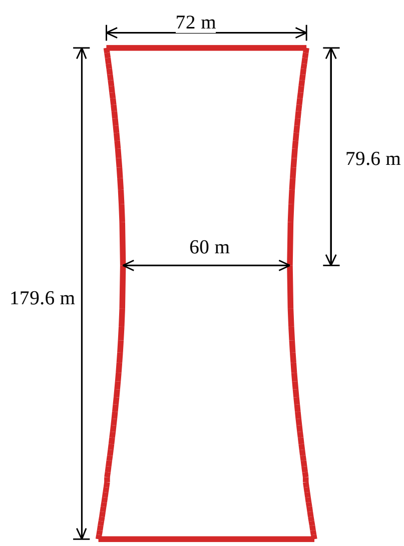


Figure 5.5. 15

Solution. We assume the center of the tower is the origin, and that the branches open to the left and right of the center. Thus, we can use the standard equation of a horizontal hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

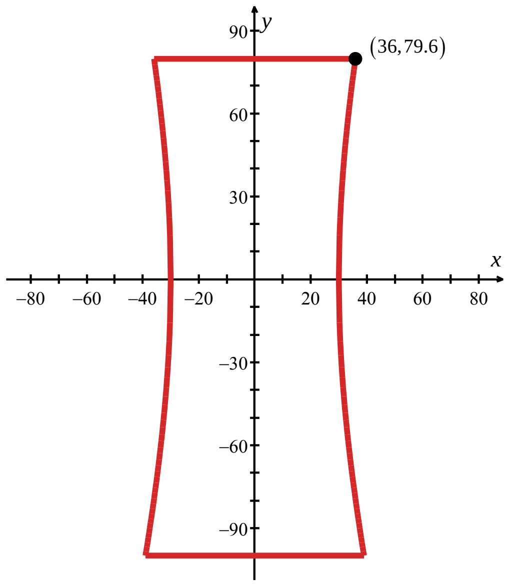


Figure 5.5. 16

We first determine a by finding the distance from the center to a vertex. Noting that the transverse axis is the *x*-axis, passing through the two sides of the cooling tower where the sides are closest, and the center $(0,0)$, the distance between vertices is 60 meters, from which $2a = 60$; so $a = 30$ and $a^2 = 900$.

To solve for b^2, we need to substitute for x and y in the equation using a known point. To do this, we can use the dimensions of the tower to find a point (x, y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the *y*-axis bisects the tower, our *x*-value can be represented by the radius of the top, or 36 meters. The *y*-value is represented by the distance from the origin to the top, which is given as 79.6 meters.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{standard equation of horizontal hyperbola centered at origin}$$

$$b^2 = \frac{y^2}{\frac{x^2}{a^2} - 1} \qquad \text{isolate } b^2$$

$$b^2 = \frac{(79.6)^2}{\frac{(36)^2}{900} - 1} \qquad \text{substitute for } a^2, x, \text{ and } y$$

$$b^2 \approx 14400.3636 \quad \text{round to four decimal places}$$

The sides of the tower can be modeled by the hyperbolic equation $\dfrac{x^2}{900} - \dfrac{y^2}{14400.3636} = 1$.

□

Identifying Conic Sections

Each of the conic sections we have studied in this chapter results from graphing equations of the general form $Ax^2 + By^2 + Cx + Dy + E = 0$ for different choices of A, B, C, D, and E, where A and B are not both zero. While we have seen examples demonstrate how to convert an equation from general form to one of the standard forms, we close this chapter with some advice about which standard form to choose.

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + By^2 + Cx + Dy + E = 0$ is a non-degenerate conic section.

If just one variable is squared, the graph is a parabola.

- Put the equation in the form of **Equation 5.4**: $(x-h)^2 = 4p(y-k)$ if there is an x^2 term, or $(y-k)^2 = 4p(x-h)$ if there is a y^2 term.

If both variables are squared, look at the coefficients of x^2 and y^2.

- If $A = B$, the graph is a circle. Put the equation in the form of **Equation 5.1**: $(x-h)^2 + (y-k)^2 = r^2$.
- If $A \neq B$ but A and B have the same sign, the graph is an ellipse. Put the equation in the form of **Equation 5.6**: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.
- If A and B have different signs, the graph is a hyperbola. Put the equation in the form of **Equation 5.9**, $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, or **Equation 5.10**, $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$.

Example 5.5.7. Identify the conic section. Write the equation in standard form.

1. $-x^2 + 2x + 3y^2 - 10 = 0$
2. $x^2 - 10x - 2y + 23 = 0$

Solution.

1. In the equation $-x^2 + 2x + 3y^2 - 10 = 0$, both variables are squared. The coefficient of x^2 is -1, which is negative, while the coefficient of y^2 is 3, which is positive. Since the coefficients have different signs, this conic section is a hyperbola. We write the equation in standard form as follows:

$$
\begin{aligned}
-x^2+2x+3y^2-10&=0\\
3y^2-x^2+2x&=10\\
3y^2-\left(x^2-2x\right)&=10\\
3y^2-\left(x^2-2x+1\right)&=10-(+1)\\
3y^2-(x-1)^2&=9\\
\frac{3y^2}{9}-\frac{(x-1)^2}{9}&=\frac{9}{9}\\
\frac{y^2}{3}-\frac{(x-1)^2}{9}&=1
\end{aligned}
$$

From the standard equation, we verify that this is a vertical hyperbola with center $(1,0)$.

2. Since only the variable x is squared in the equation $x^2-10x-2y+23=0$, this is the equation of a parabola. We rewrite the equation in the standard form of a parabola that has an x^2 term.

$$
\begin{aligned}
x^2-10x-2y+23&=0\\
x^2-10x&=2y-23\\
\left(x^2-10x+25\right)&=2y-23+25\\
(x-5)^2&=2y+2\\
(x-5)^2&=2(y+1)
\end{aligned}
$$

Having the equation in standard form, it is easy to identify this parabola as having a vertex of $(5,-1)$ and opening upward.

□

5.5 Exercises

1. Define a hyperbola in terms of its foci.

2. What can we conclude about a hyperbola if its asymptotes intersect at the origin?

3. If the transverse axis of a hyperbola is vertical, what do we know about the graph?

In Exercises 4 – 8, write the equation in standard form if it represents a hyperbola. If the equation does not represent a hyperbola, explain how the equation violates the definition of a hyperbola.

4. $3y^2 + 2x = 6$

5. $\frac{x^2}{36} - \frac{y^2}{9} = 1$

6. $5y^2 + 4x^2 = 6x$

7. $25x^2 - 16y^2 = 400$

8. $-9x^2 + 18x + y^2 + 4y - 14 = 0$

In Exercises 9 – 20, find the center, the vertices, and the foci of the hyperbola. Find the equations of the asymptotes and graph the hyperbola.

9. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

10. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

11. $\frac{x^2}{49} - \frac{y^2}{16} = 1$

12. $\frac{y^2}{9} - \frac{x^2}{25} = 1$

13. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

14. $\frac{(y-3)^2}{11} - \frac{(x-1)^2}{10} = 1$

15. $\frac{(x+4)^2}{16} - \frac{(y-4)^2}{1} = 1$

16. $\frac{(x+1)^2}{9} - \frac{(y-3)^2}{4} = 1$

17. $\frac{(y+2)^2}{16} - \frac{(x-5)^2}{20} = 1$

18. $\frac{(x-4)^2}{8} - \frac{(y-2)^2}{18} = 1$

19. $\frac{(y+5)^2}{9} - \frac{(x-4)^2}{25} = 1$

20. $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$

In Exercises 21 – 30, put the equation of the hyperbola into standard form. Find the center, the vertices, and the foci. Graph the hyperbola.

21. $12x^2 - 3y^2 + 30y - 111 = 0$

22. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$

23. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$

24. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$

25. $-4x^2 + 16y^2 - 8x - 32y - 52 = 0$

26. $x^2 - 25y^2 - 8x - 100y - 109 = 0$

27. $-x^2 + 4y^2 + 8x - 40y + 88 = 0$

28. $64x^2 - 9y^2 + 128x - 72y - 656 = 0$

29. $16x^2 - 4y^2 + 64x - 8y - 4 = 0$

30. $-100x^2 + y^2 + 1000x - 10y - 2575 = 0$

In Exercises 31 – 42, find the standard form of the equation of the hyperbola that has the given properties.

31. Vertices $(0, \pm 5)$, Foci $(0, \pm 8)$

32. Vertices $(\pm 3, 0)$, Focus $(5, 0)$

33. Vertices $(0, \pm 6)$, Focus $(0, -8)$

34. Center $(0, 0)$, Vertex $(0, -13)$, Focus $\left(0, \sqrt{313}\right)$

35. Foci $(\pm 5, 0)$; Segment of the conjugate axis bordered by the guide rectangle has length 6.

36. Center $(3, 7)$, Vertex $(3, 3)$, Focus $(3, 2)$

37. Center $(4, 2)$, Vertex $(9, 2)$, Focus $\left(4 + \sqrt{26}, 2\right)$

38. Center $(3, 5)$, Vertex $(3, 11)$, Focus $\left(3, 5 + 2\sqrt{10}\right)$

39. Vertices $(0, 1)$ and $(8, 1)$, Focus $(-3, 1)$

40. Vertices $(1, 1)$ and $(11, 1)$, Focus $(12, 1)$

41. Vertices $(3, 2)$ and $(13, 2)$; Guide rectangle intersects conjugate axis at $(8, 4)$ and $(8, 0)$.

42. Vertex $(-10, 5)$, Asymptotes $y = \pm \frac{1}{2}(x - 6) + 5$

In Exercises 43 – 47, given the graph of the hyperbola, determine its equation.

43.

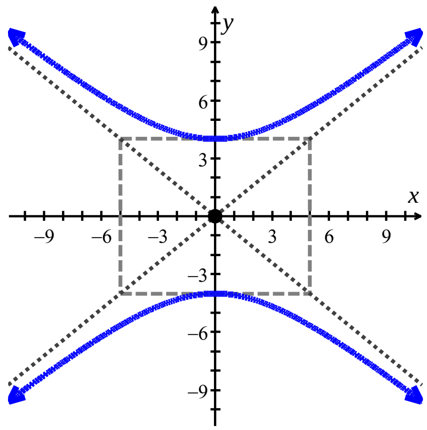


Figure Ex5.5. 1

44.

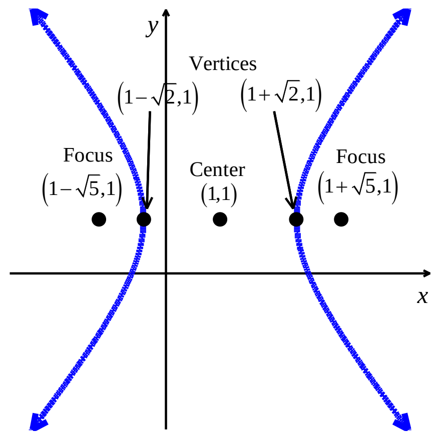


Figure Ex5.5. 2

45.

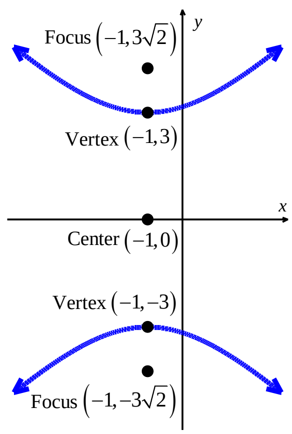


Figure Ex5.5. 3

46.

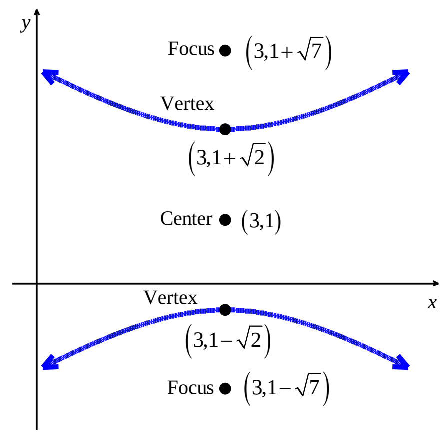


Figure Ex5.5. 4

47.

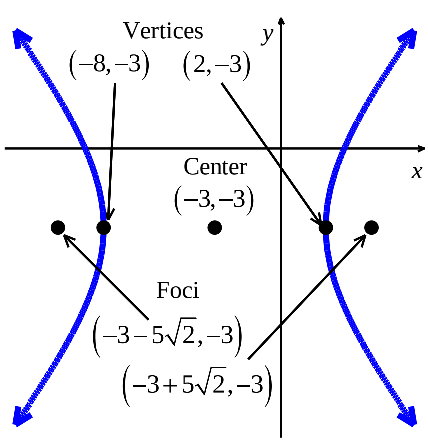


Figure Ex5.5. 5

48. The design layout of a hyperbolic cooling tower is shown below. The tower stands 167.082 meters tall. The diameter of the top is 60 meters. At their closest, the sides of the tower are 40 meters apart. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola is the origin of the coordinate plane. Round final values to four decimal places.

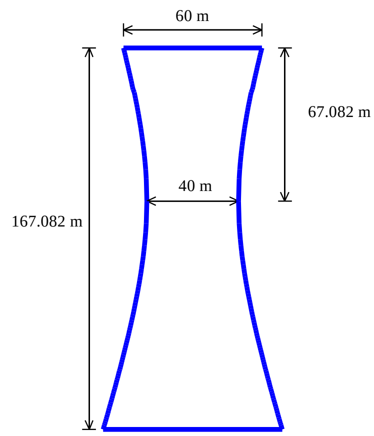


Figure Ex5.5. 6

49. The cross section of a hyperbolic cooling tower is shown below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground). Determine the height of the tower to the nearest foot.

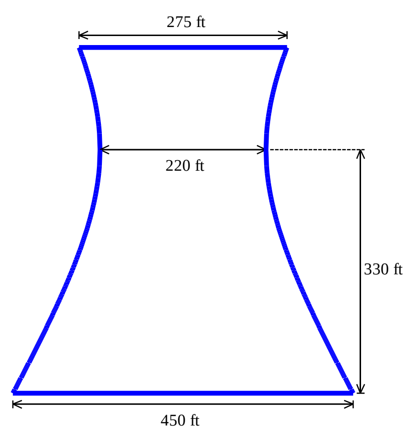


Figure Ex5.5. 7

50. A hedge is to be constructed in the shape of a hyperbola near a fountain at the center of a yard. The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is 5 yards. Find the equation of the hyperbola and sketch the graph.

51. A hedge is to be constructed in the shape of a hyperbola near a fountain at the center of a yard. The hedge will follow the asymptotes $y=\frac{3}{4}x$ and $y=-\frac{3}{4}x$, and its closest distance to the center fountain is 20 yards. Find the equation of the hyperbola and sketch the graph.

52. With the help of your classmates, show that if $Ax^2+By^2+Cx+Dy+E=0$ determines a non-degenerate conic[15] then

- $AB<0$ means that the graph is a hyperbola
- $AB=0$ means that the graph is a parabola
- $AB>0$ means that the graph is an ellipse or circle

In Exercises 53 – 62, find the standard form of the equation using the Strategies for Identifying Conic Sections and then graph any resulting conic sections.

53. $x^2-2x-4y-11=0$
54. $x^2+y^2-8x+4y+11=0$
55. $9x^2+4y^2-36x+24y+36=0$
56. $9x^2-4y^2-36x-24y-36=0$
57. $y^2+8y-4x+16=0$
58. $4x^2+y^2-8x+4=0$
59. $4x^2+9y^2-8x+54y+49=0$
60. $x^2+y^2-6x+4y+14=0$
61. $2x^2+4y^2+12x-8y+25=0$
62. $4x^2-5y^2-40x-20y+160=0$

63. The points $(-4,2)$, $(2,2)$, $\left(\frac{-3\sqrt{29}-5}{5},0\right)$, $\left(\frac{3\sqrt{29}-5}{5},0\right)$, $\left(4,-\frac{14}{3}\right)$ and $\left(4,\frac{26}{3}\right)$ lie on the hyperbola $\frac{(x+1)^2}{9}-\frac{(y-2)^2}{25}=1$. Find three other points that lie on the hyperbola.

[15] Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

CHAPTER 6
SYSTEMS OF EQUATIONS AND MATRICES

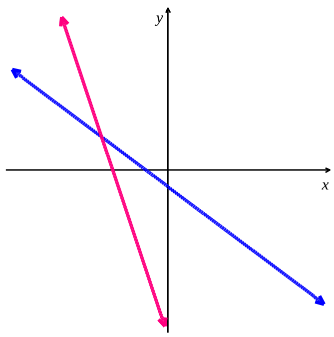


Figure 6.0. 1

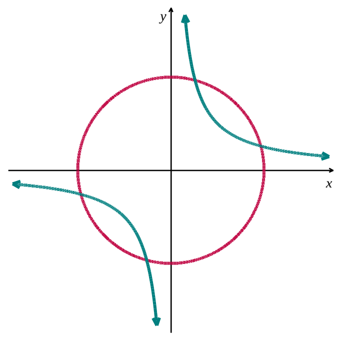


Figure 6.0. 2

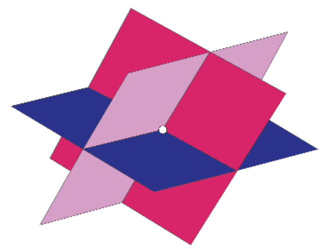


Figure 6.0. 3

Chapter Outline

Introduction

In this chapter we extend ideas for solving 2×2 (two-by-two) systems of linear equations to solving 2×2 nonlinear systems, systems of 3×3 (three-by-three) linear equations, and developing ideas about augmented matrix systems. We then transition to matrix arithmetic where you will see that it shares some properties with real number arithmetic; most importantly, some square matrices have inverses and inverses may be used to solve matrix equations (in much the same way we use inverse operations in arithmetic to solve algebraic equations.) Additionally, you will learn a little about determinants of square matrices, what they tell you about a system, and how they may be used to solve systems.

Section 6.1 starts with a review of solving 2×2 systems of linear equations. The goal of the review is to remind you of both the graphical meaning of a solution and the two basic analytic processes for finding solutions (substitution and elimination.) We build on these analytic processes to solve systems of nonlinear equations. In addition, it will be helpful to think about the graphs of equations in the nonlinear systems. For example, if you are asked to find the point(s) of intersection for a circle and a parabola, it will be helpful to visualize all the possibilities of their intersections (0, 1, 2, 3, or 4 points of intersection in this case.) Analytically, as with systems of linear equations, you will start by finding solution(s) for one of the variables and then substituting it/them into an equation to find the corresponding value(s) for the other variable.

Section 6.2 also builds on your previous understanding for solving 2×2 systems, but now with 3×3 systems. Again, you will be looking for intersections; geometrically this means looking for intersections amongst three planes. Analytically, the process is very much like the elimination and substitution processes for 2×2 systems, but with an extra variable. This process will lead to a systematic method of solution in the next section.

In Section 6.3, you will learn how to simplify the process of Section 6.2 by using augmented matrices with a process called Gaussian Elimination. The method allows you to get the system in row-echelon and/or reduced row-echelon form. Although, in general, there are many ways to obtain these forms, we will develop a standard method.

In Section 6.4 you are introduced to matrix operations, namely addition, subtraction, scalar multiplication, and matrix multiplication. Further, you will see that for matrix addition, the commutative, associative, identity, and inverse properties hold. For matrix multiplication, associativity holds. There is an identity matrix for multiplication, and the distributive property holds. Arithmetic manipulations with matrices will be useful for solving systems in this chapter and in future mathematics courses.

In Section 6.5 you will learn to find inverses of 2×2 and 3×3 invertible matrices and use inverses to solve linear systems. Section 6.6 progresses with the ideas in Section 6.5 by introducing determinants, how to find them for 2×2 and 3×3 matrices, and then how to use them for solving linear systems using Cramer's Rule. In the example with the 2×2 systems, you will see that if the two lines represented by the equations are parallel, the determinant will be zero and the method of solution by inverse matrices or determinants fails.

Section 6.7 deals with the decomposition of fractions. In this section, you will learn to write a rational expression as the sum of two or more simpler expressions. This process relies on the skills you learned in this chapter and will be helpful when you take Calculus.

6.1 Systems of Linear and Nonlinear Equations

Learning Objectives

- Solve systems of two linear equations in two variables using substitution.
- Solve systems of two linear equations in two variables using elimination.
- Interpret solutions to 2×2 systems of linear equations.
- Solve systems of two nonlinear equations in two variables using elimination.
- Solve systems of two nonlinear equations in two variables using substitution.
- Interpret solutions to 2×2 systems of nonlinear equations.

We begin our study of systems of equations by reviewing the definition of a linear equation in two variables.

Definition 6.1. A **linear equation in two variables** x and y is an equation of the form $ax+by=c$ where a, b, and c are real numbers and at least one of a and b is nonzero.

The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations that are nonlinear are $x^2+y=1$, $xy=5$ and $e^{2x}+\ln(y)=1$. We leave it to the reader to explain why these equations do not satisfy **Definition 6.1**.

From graphing linear equations in prior mathematics classes, you will recall that the graph of a linear equation is a line. If we couple two linear equations, in two variables, together in order to find the points of intersection of two lines, we obtain a **2×2 system of linear equations**. We read '2×2' as 'two by two'. The first 2 designates the number of equations and the second 2 is the number of variables.

Solving 2×2 Systems of Linear Equations

While the following examples may be review, they provide a good starting place for our study of systems of equations and matrices.

Example 6.1.1. Use the substitution method to solve the system of equations.

$$\begin{cases} 2x-y=1 \\ -4x+y=-5 \end{cases}$$

Solution. We solve the second equation for y in terms of x.

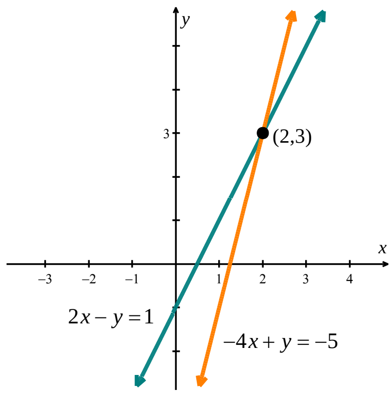
$$-4x + y = -5$$
$$y = 4x - 5$$

We **substitute** this expression, $y = 4x - 5$, for y in the first equation.

$$\begin{aligned} 2x - y &= 1 && \text{1st equation} \\ 2x - (4x - 5) &= 1 && \text{substitute } y = 4x - 5 \text{ from 2nd equation} \\ 2x - 4x + 5 &= 1 \\ -2x &= -4 \end{aligned}$$

We find $x = 2$, and we use this result along with the equation $y = 4x - 5$ to get $y = 4(2) - 5 = 3$. The solution to the system is $x = 2$ and $y = 3$. As shown in the following graph, this solution, stated as the ordered pair $(x, y) = (2, 3)$, is the point of intersection of the lines $2x - y = 1$ and $-4x + y = -5$.

Figure 6.1. 1



□

We may also check solutions algebraically by substituting the *x*- and *y*-values in the solution into the original equations to see that they are satisfied.

Example 6.1.2. Use the elimination method to solve the following systems of equations.

1. $\begin{cases} 3x + 4y = -2 \\ -3x - y = 5 \end{cases}$ 2. $\begin{cases} 2x - 4y = 6 \\ 3x - 6y = 9 \end{cases}$ 3. $\begin{cases} 6x + 3y = 9 \\ 4x + 2y = 12 \end{cases}$

Solution.

1. To solve the system of equations, we use **addition** to **eliminate** the variable x. We take the two equations as given and add them together.

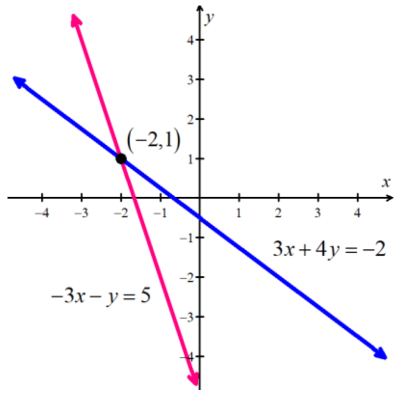
$$\begin{array}{r} 3x + 4y = -2 \\ -3x - y = 5 \\ \hline 3y = 3 \end{array}$$

This gives us $y = 1$, which we may substitute in either of the two equations to solve for x. We select the first equation, $3x + 4y = -2$.

$$\begin{aligned} 3x+4(1) &= -2 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

Our solution is $(x, y) = (-2, 1)$. Below, the graphs of the two equations verify that $(-2,1)$ is the point of intersection.

Figure 6.1. 2

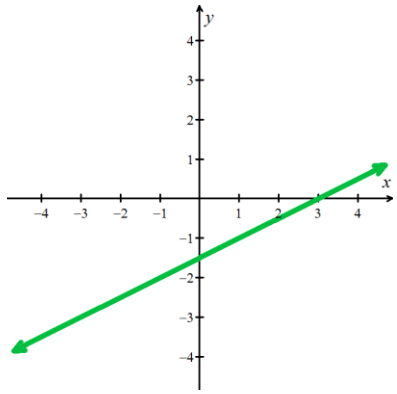


2. Adding the two equations, $2x-4y=6$ and $3x-6y=9$, directly fails to eliminate either of the variables, but we note that if we multiply both sides of the first equation by 3 and both sides of the second equation by −2, we will be in a position to eliminate the *x*-term.

$$\begin{aligned} 6x-12y &= 18 \\ \underline{-6x+12y} &\underline{= -18} \\ 0 &= 0 \end{aligned}$$

We eliminated not only the *x*-term, but the *y*-term as well, and we are left with the identity $0=0$. This means that the two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x, y) satisfies the equation $2x-4y=6$, it automatically satisfies the equation $3x-6y=9$. One way to describe the solution set to this system, using set-builder notation, is $\{(x, y) \mid 2x-4y=6\}$. Geometrically, $2x-4y=6$ and $3x-6y=9$ are the same line, which means that they intersect at every point on their graphs.

Figure 6.1. 3



$$2x - 4y = 6$$
$$3x - 6y = 9$$

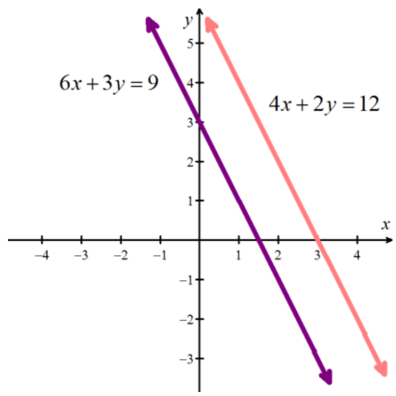
(same line)

3. Multiplying both sides of the first equation, $6x+3y=9$, by 2 and both sides of the second equation, $4x+2y=12$, by −3, we set the stage to eliminate x.

$$\begin{array}{r} 12x+6y = \ \ 18 \\ \underline{-12x-6y = -36} \\ 0 \ = -18 \end{array}$$

As in the previous problem, both x and y dropped out of the equation, but we are left with an irrevocable contradiction, $0=-18$. This tells us that it is impossible to find a pair (x, y) that satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x+3y=9$ and $4x+2y=12$ are distinct and parallel, so they do not intersect. Note that if two lines have the same slope they will either represent the same line as in part 2, or they will be parallel as in this example.

Figure 6.1. 4



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If a system of equations, linear or nonlinear, has no solutions it is called **inconsistent**. Systems, linear or nonlinear, with at least one solution are called **consistent**. The systems in **Example 6.1.1** and **Example 6.1.2**, parts 1 and 2, are consistent. The system in **Example 6.1.2**, part 3, is inconsistent.

In the case of linear systems, we can divide the consistent systems into two categories. A consistent linear system is called **independent** if it has a unique solution, as in **Example 6.1.1** and **Example 6.1.2**, part 1. Geometrically, this solution is the point of intersection of two lines having different slopes. A consistent linear system is called **dependent** if it has an infinite number of solutions, as the system in **Example 6.1.2**, part 2. Geometrically, the lines are coincident and every ordered pair on the line is a solution to both equations.

We move on to the challenge of solving systems of nonlinear equations, in which we apply the above techniques while addressing additional issues such as extraneous solutions.

Solving 2×2 Systems of Nonlinear Equations

By now, we have seen many nonlinear equations in two variables, such as polynomial equations. These equations represent curves in the plane. Combining more than one nonlinear equation in two variables results in a **system of nonlinear equations in two variables**. A system with two nonlinear equations in two variables is referred to as a **2×2 system of nonlinear equations**. Such a system can be used to find the points of intersection of the two curves represented by the equations. There is no general method for solving nonlinear systems. We will try to use the substitution method or the elimination method, or a combination, to reduce the system to one equation in one variable.

Example 6.1.3. Solve the following systems of equations.

1. $\begin{cases} x^2+y^2=4 \\ 4x^2+9y^2=36 \end{cases}$

2. $\begin{cases} x^2+y^2=4 \\ 4x^2-9y^2=36 \end{cases}$

Solution. We can solve each system by either substitution or elimination. To demonstrate both, we will solve the first system by the substitution method and the second by the elimination method.

1. We can solve for x^2 in the first equation to get $x^2=4-y^2$, and substitute this result in the second equation to find *y*-value(s) at the point(s) of intersection.

$$\begin{aligned} 4x^2+9y^2&=36 \\ 4\left(4-y^2\right)+9y^2&=36 \\ 16-4y^2+9y^2&=36 \\ 5y^2&=20 \end{aligned}$$

We find $y^2=4$, from which $y=\pm\,2$. To determine the associated *x*-values, we substitute each value of y into one of the original equations. Here, we choose $x^2+y^2=4$.

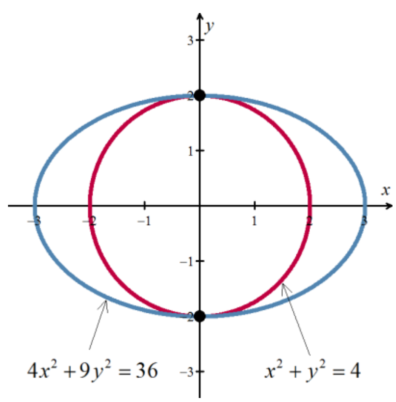
$$x^2+(-2)^2=4 \quad \text{set } y=-2 \qquad\qquad x^2+(2)^2=4 \quad \text{set } y=2$$
$$x^2=0 \qquad\qquad\qquad\qquad x^2=0$$
$$x=0 \qquad\qquad\qquad\qquad x=0$$

The potential solutions are the ordered pairs $(0,-2)$ and $(0,2)$. We check each ordered pair algebraically; to be a solution it must satisfy both of the original equations.

(x,y)	First equation: $x^2+y^2=4$	Second equation: $4x^2+9y^2=36$
$(0,-2)$	Left side is $(0)^2+(-2)^2=4$	Left side is $4(0)^2+9(-2)^2=36$.
$(0,2)$	Left side is $(0)^2+(2)^2=4$.	Left side is $4(0)^2+9(2)^2=36$

For each potential solution, substituting the values of the ordered pair in the left side of each equation gives the value of the right side of the equation, so both potential solutions are acceptable. To visualize the solutions, we sketch both equations and look for their points of intersection. The graph of $x^2+y^2=4$ is a circle centered at $(0,0)$ with a radius of 2, whereas the graph of $4x^2+9y^2=36$, when written in the standard form $\frac{x^2}{9}+\frac{y^2}{4}=1$, is easily recognized as an ellipse centered at $(0,0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4.

Figure 6.1. 5



We see from the graph that the two curves intersect only at their y-intercepts, $(0,-2)$ and $(0,2)$.

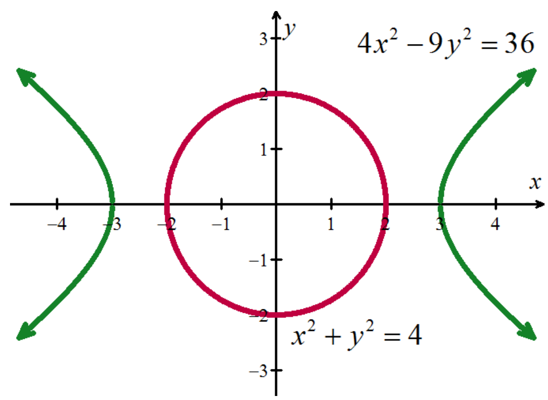
We have verified, both algebraically and geometrically, that the solution set is $\{(0,-2),(0,2)\}$.

2. Using the elimination method, we multiply the equation $x^2+y^2=4$ by -4, then add the result to $4x^2-9y^2=36$.

$$\begin{aligned} -4x^2 - 4y^2 &= -16 \\ \underline{4x^2 - 9y^2} &\underline{= \ \ 36} \\ -13y^2 &= \ \ 20 \end{aligned}$$

The equation $y^2 = -\dfrac{20}{13}$ does not have any real solutions since y^2 must be greater than or equal to zero. Therefore, this system is inconsistent. There is no solution. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$, we see that it represents a hyperbola centered at $(0,0)$, opening to the left and right with vertices $(\pm 3, 0)$.

Figure 6.1. 6



The circle and the hyperbola have no points in common.

□

Following is a strategy for solving a system of two nonlinear equations in two variables.

Solving a 2×2 System of Nonlinear Equations

1. Apply substitution, elimination, or a combination, to obtain equation(s) in one variable.
   a) In applying substitution, you may solve an equation for a variable, an expression, or a combination of variables.
   b) In applying elimination, you may also multiply or divide equations by variables.
2. Solve the new equation(s).
3. Substitute solutions from part 2, if any, in one of original equations to obtain corresponding values of the other variable.
4. Check each potential ordered pair solution obtained in part 3.

If there is no solution in part 2, the system has no solution.[1] Recall that such a system is called inconsistent. Graphically, this means the two curves do not intersect. We may have one or more solutions after checking the potential solutions in part 4, in which case you may recall that such a system is called consistent. Each solution corresponds to a point of intersection of the two curves.

Checking of potential solutions is not necessary for every nonlinear system. However, rather than specifying the type of nonlinear systems, or instances, for which checking potential solutions is essential, we will check all potential solutions.

Example 6.1.4. Solve the system of equations.

$$\begin{cases} x^2 + y^2 = 5 \\ y - 2x = 0 \end{cases}$$

Solution. We observe that it is easy to solve for one of the variables in the second equation, while it is not obvious how to combine the two equations to eliminate a variable, so we will apply the substitution method. We begin by solving $y - 2x = 0$ to get $y = 2x$, and substitute $y = 2x$ in $x^2 + y^2 = 5$ to get $x^2 + (2x)^2 = 5$. We proceed by solving for *x*.

$$\begin{aligned} x^2 + (2x)^2 &= 5 \\ 5x^2 &= 5 \\ x^2 &= 1 \end{aligned}$$

We find $x = \pm 1$, which we substitute in the first equation, $x^2 + y^2 = 5$, to determine corresponding *y*-values.

$$\begin{aligned} (-1)^2 + y^2 &= 5 \quad \text{input } x = -1 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned} \qquad\qquad \begin{aligned} (1)^2 + y^2 &= 5 \quad \text{input } x = 1 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

Our potential solutions are the ordered pairs, $(-1,-2)$, $(-1,2)$, $(1,-2)$, and $(1,2)$. We check each of these potential solutions. Since we already know they satisfy the first equation, we only need to check them in the second equation, $y - 2x = 0$.

[1] We may also say the solution set is the empty set.

(x, y)	Second Equation: $y - 2x = 0$
$(-1,-2)$	Left side is $(-2)-2(-1)=0$, so left side equals right side.
$(-1,2)$	Left side is $(2)-2(-1)=4$, so left side is NOT equal to right side.
$(1,-2)$	Left side is $(-2)-2(1)=-4$, so left side is NOT equal to right side.
$(1,2)$	Left side is $(2)-2(1)=0$, so left side equals right side.

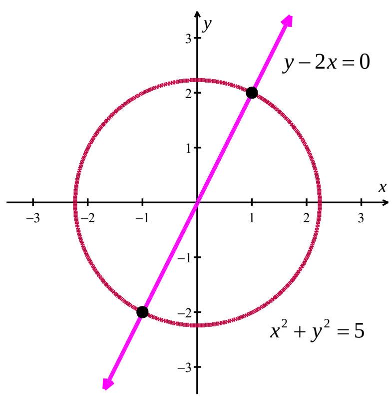
Both $(-1,2)$ and $(1,-2)$ are extraneous solutions. The solution set is $\{(-1,-2),(1,2)\}$.

□

In the previous example, if we had plugged $x = \pm 1$ into the second equation, $y - 2x = 0$, to find *y*-values, we would not have had any extraneous solutions. It is not always easy to tell which scenario will result in extraneous solutions, but checking all potential solutions eliminates the necessity for that determination.

To visualize the solutions in **Example 6.1.4**, we graph the circle $x^2 + y^2 = 5$ and the line $y - 2x = 0$ below. Their two points of intersection are the two solutions.

Figure 6.1. 7



There is no general method for solving nonlinear systems, and at times it may take several steps to obtain an equation in one variable. Some creativity may be helpful, as demonstrated in the following examples.

Example 6.1.5. Solve the system of equations.

$$\begin{cases} x^2 + x - y = 0 \\ \dfrac{y^2}{x} - \dfrac{y}{x} + 1 = 0 \end{cases}$$

Solution. We use substitution by solving the first equation for y to get $y = x^2 + x$, and then substitute this result in the second equation: $\frac{\left(x^2+x\right)^2}{x} - \frac{x^2+x}{x} + 1 = 0$. Now that we have an equation in one variable, we can solve this equation for x by, first of all, multiplying both sides by x.

$$\begin{aligned} \left(x^2+x\right)^2 - \left(x^2+x\right) + x &= 0 \\ x^4 + 2x^3 + x^2 - x^2 - x + x &= 0 \\ x^4 + 2x^3 &= 0 \\ x^3\left(x+2\right) &= 0 \end{aligned}$$

We get $x^3 = 0$ or $x + 2 = 0$, resulting in $x = 0$ or $x = -2$. We substitute each of these x-values in the first equation, $x^2 + x - y = 0$, to find corresponding y-values.

$$\begin{aligned} (0)^2 + (0) - y &= 0 \quad \text{input } x = 0 \\ 0 &= y \\ y &= 0 \end{aligned} \qquad \begin{aligned} (-2)^2 + (-2) - y &= 0 \quad \text{input } x = -2 \\ 4 - 2 &= y \\ y &= 2 \end{aligned}$$

Our potential solutions are $(0,0)$ and $(-2,2)$. We know these potential solutions satisfy the first equation, and proceed by checking that they also satisfy the second equation.

(x,y)	Second Equation: $\frac{y^2}{x} - \frac{y}{x} + 1 = 0$
$(0,0)$	Left side is $\frac{(0)^2}{(0)} - \frac{(0)}{(0)} + 1$, which is undefined due to zero in denominator.
$(-2,2)$	Left side is $\frac{(2)^2}{(-2)} - \frac{(2)}{(-2)} + 1 = -2 + 1 + 1 = 0$, so left side equals right side.

The only solution is $(x,y) = (-2,2)$.

Alternate Solution. An alternate way to arrive at the x-values of 0 and -2 in **Example 6.1.5** is to begin by multiplying the second equation, $\frac{y^2}{x} - \frac{y}{x} + 1 = 0$, through by $-x$. We then have the system

$$\begin{cases} x^2 + x - y = 0 \\ -y^2 + y - x = 0 \end{cases}$$

Adding these two equations gives us $x^2 - y^2 = 0$, which we solve by factoring as $(x-y)(x+y) = 0$ to get $y = x$ or $y = -x$. We substitute these results into the first equation, $x^2 + x - y = 0$.

$$
\begin{aligned}
x^2+x-(-x)&=0 \quad \text{substitute } y=-x \\
x^2+2x&=0 \\
x(x+2)&=0
\end{aligned}
\qquad\qquad
\begin{aligned}
x^2+x-(x)&=0 \quad \text{substitute } y=x \\
x^2&=0 \\
x&=0
\end{aligned}
$$

We find $x=0$ or $x=-2$ and proceed to solve for y as in the original solution.

□

Example 6.1.6. Solve the system of equations.

$$
\begin{cases} x^2-3xy+2y^2=0 \\ x^2+xy=6 \end{cases}
$$

Solution. It is not obvious how to eliminate a variable so we will try the substitution method. Although solving for one variable in terms of the other is not immediate, we can begin by factoring the first equation.

$$
\begin{aligned}
x^2-3xy+2y^2&=0 \\
(x-y)(x-2y)&=0
\end{aligned}
$$

Setting each factor equal to zero, we get $y=x$ or $y=\frac{x}{2}$. We plug each of these into the second equation, $x^2+xy=6$, then solve for x.

$$
\begin{aligned}
x^2+x(x)&=6 \quad \text{substitute } y=x \\
2x^2&=6 \\
x^2&=3
\end{aligned}
\qquad\qquad
\begin{aligned}
x^2+x\left(\frac{x}{2}\right)&=6 \quad \text{substitute } y=\frac{x}{2} \\
\frac{3}{2}x^2&=6 \\
x^2&=4
\end{aligned}
$$

The result is $x=\pm\sqrt{3}$ or $x=\pm 2$. Now we use the second equation, $x^2+xy=6$, to solve for y-values.

Input $x=-\sqrt{3}$:

$$
\begin{aligned}
\left(-\sqrt{3}\right)^2+\left(-\sqrt{3}\right)y&=6 \\
3-\sqrt{3}y&=6 \\
y&=\frac{3}{-\sqrt{3}} \\
y&=-\sqrt{3}
\end{aligned}
$$

Input $x=\sqrt{3}$:

$$
\begin{aligned}
\left(\sqrt{3}\right)^2+\left(\sqrt{3}\right)y&=6 \\
3+\sqrt{3}y&=6 \\
y&=\frac{3}{\sqrt{3}} \\
y&=\sqrt{3}
\end{aligned}
$$

Input $x=-2$:

$$
\begin{aligned}
(-2)^2+(-2)y&=6 \\
4-2y&=6 \\
y&=\frac{2}{-2} \\
y&=-1
\end{aligned}
$$

Input $x=2$:

$$
\begin{aligned}
(2)^2+(2)y&=6 \\
4+2y&=6 \\
y&=\frac{2}{2} \\
y&=1
\end{aligned}
$$

We have the potential solutions $\left(-\sqrt{3},-\sqrt{3}\right)$, $\left(\sqrt{3},\sqrt{3}\right)$, $(-2,-1)$ and $(2,1)$. We check each of these in the first equation, $x^2-3xy+2y^2=0$, since we already know they satisfy the second.

(x, y)	First Equation: $x^2 - 3xy + 2y^2 = 0$
$\left(-\sqrt{3}, -\sqrt{3}\right)$	Left Side is $\left(-\sqrt{3}\right)^2 - 3\left(-\sqrt{3}\right)\left(-\sqrt{3}\right) + 2\left(-\sqrt{3}\right)^2 = 3 - 9 + 6 = 0$, same as right side.
$\left(\sqrt{3}, \sqrt{3}\right)$	Left side is $\left(\sqrt{3}\right)^2 - 3\left(\sqrt{3}\right)\left(\sqrt{3}\right) + 2\left(\sqrt{3}\right)^2 = 3 - 9 + 6 = 0$, same as right side.
$(-2, -1)$	Left side is $(-2)^2 - 3(-2)(-1) + 2(-1)^2 = 4 - 6 + 2 = 0$, same as right side.
$(2, 1)$	Left side is $(2)^2 - 3(2)(1) + 2(1)^2 = 4 - 6 + 2 = 0$, same as right side.

There are four solutions: $\left(-\sqrt{3}, -\sqrt{3}\right)$, $\left(\sqrt{3}, \sqrt{3}\right)$, $(-2, -1)$ and $(2, 1)$.

$\square$

6.1 Exercises

1. Can a system of linear equations have exactly two solutions? Explain why or why not.

2. Can a system of two non-linear equations have exactly two solutions? What about exactly three? In not, explain why not. If so, sketch the graph of such a system.

In Exercises 3 – 16, solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent.

3. $\begin{cases} x+2y=5 \\ x=6 \end{cases}$

4. $\begin{cases} 2y-3x=1 \\ y=-3 \end{cases}$

5. $\begin{cases} x+3y=5 \\ 2x+3y=4 \end{cases}$

6. $\begin{cases} x-2y=3 \\ -3x+6y=-9 \end{cases}$

7. $\begin{cases} 3x-2y=18 \\ 5x+10y=-10 \end{cases}$

8. $\begin{cases} 4x+2y=-10 \\ 3x+9y=0 \end{cases}$

9. $\begin{cases} -2x+5y=-42 \\ 7x+2y=30 \end{cases}$

10. $\begin{cases} 6x-5y=-34 \\ 2x+6y=4 \end{cases}$

11. $\begin{cases} -x+2y=-1 \\ 5x-10y=6 \end{cases}$

12. $\begin{cases} 5x+9y=16 \\ x+2y=4 \end{cases}$

13. $\begin{cases} \dfrac{x+2y}{4}=-5 \\ \dfrac{3x-y}{2}=1 \end{cases}$

14. $\begin{cases} \dfrac{1}{2}x-\dfrac{1}{3}y=-1 \\ 2y-3x=6 \end{cases}$

15. $\begin{cases} x+4y=6 \\ \dfrac{1}{12}x+\dfrac{1}{3}y=\dfrac{1}{2} \end{cases}$

16. $\begin{cases} 3y-\dfrac{3}{2}x=-\dfrac{15}{2} \\ \dfrac{1}{2}x-y=\dfrac{3}{2} \end{cases}$

In Exercises 17 – 20, graph the system of equations and state whether the system has one solution, no solution, or infinite solutions.

17. $\begin{cases} -x+2y=4 \\ 2x-4y=1 \end{cases}$

18. $\begin{cases} x+2y=7 \\ 2x+6y=12 \end{cases}$

19. $\begin{cases} 3x-5y=7 \\ x-2y=3 \end{cases}$

20. $\begin{cases} 3x-2y=5 \\ -9x+6y=-15 \end{cases}$

In Exercises 21 – 32, solve the given system of non-linear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

21. $\begin{cases} x+y=4 \\ x^2+y^2=9 \end{cases}$

22. $\begin{cases} y=x-3 \\ x^2+y^2=9 \end{cases}$

23. $\begin{cases} y=x \\ x^2+y^2=9 \end{cases}$

24. $\begin{cases} y=-x \\ x^2+y^2=9 \end{cases}$

25. $\begin{cases} x=2 \\ x^2-y^2=9 \end{cases}$

26. $\begin{cases} 4x^2-9y^2=36 \\ 4x^2+9y^2=36 \end{cases}$

27. $\begin{cases} x^2+y^2=25 \\ x^2-y^2=1 \end{cases}$

28. $\begin{cases} x^2-y=4 \\ x^2+y^2=4 \end{cases}$

29. $\begin{cases} x^2+y^2=4 \\ x^2-y=5 \end{cases}$

30. $\begin{cases} x^2+y^2=16 \\ 16x^2+4y^2=64 \end{cases}$

31. $\begin{cases} x^2+y^2=16 \\ 9x^2-16y^2=144 \end{cases}$

32. $\begin{cases} x^2+y^2=16 \\ \dfrac{1}{9}y^2-\dfrac{1}{16}x^2=1 \end{cases}$

In Exercises 33 – 40, solve the given system of non-linear equations.

33. $\begin{cases} x^2+y^2=16 \\ x-y=2 \end{cases}$

34. $\begin{cases} x^2-y^2=1 \\ x^2+4y^2=4 \end{cases}$

35. $\begin{cases} x+2y^2=2 \\ x^2+4y^2=4 \end{cases}$

36. $\begin{cases} (x-2)^2+y^2=1 \\ x^2+4y^2=4 \end{cases}$

37. $\begin{cases} x^2+y^2=25 \\ y-x=1 \end{cases}$

38. $\begin{cases} x^2+y^2=25 \\ x^2+(y-3)^2=10 \end{cases}$

39. $\begin{cases} y=x^3+8 \\ y=10x-x^2 \end{cases}$

40. $\begin{cases} x^3-10x+y=5 \\ x-y=-5 \end{cases}$

6.2 Systems of Linear Equations and Applications

Learning Objectives

- Solve systems of three linear equations in three variables.
- Interpret solutions to 3×3 systems of linear equations.
- Solve application problems modeled by linear equations in three variables.

We begin this section with the definition of a linear equation in three variables.

Definition 6.2. A **linear equation in three variables** x, y, and z is an equation of the form $ax+by+cz=d$ where a, b, c, and d are real numbers and at least one of a, b, and c is nonzero.

Just as equations involving the variables x and y describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving x, y, and z describe objects called **surfaces** in three-dimensional space. Linear equations, as described above, represent planes in three-dimensional space. Combining more than one linear equation in three variables results in a **system of linear equations in three variables**. A system with three linear equations in three variables is called a **3×3 system of linear equations**, and is used to find the points of intersection of the three planes represented by the equations.

Solving 3×3 Systems of Linear Equations

When solving systems of equations in more than two variables, it becomes increasingly important to keep track of which operations are performed to which equations and to develop a strategy based on the manipulations we have already employed. To this end, we identify a strategy that can be used in solving 3×3 systems of linear equations.

Solving a 3×3 System of Linear Equations

1. Target one of the three variables for elimination.
2. Combine a pair of equations to eliminate the targeted variable, resulting in a single equation that does not contain the variable.
3. Combine another pair of equations to eliminate the targeted variable, resulting in a single equation that does not contain the variable.
4. Solve the resulting 2×2 system for both variables.
5. Back-substitute the values from step 4 into one of the original equations to find the value of the targeted variable.

The idea of systems being consistent or inconsistent carries over from **Section 6.1**. We say a 3×3 system of linear equations is inconsistent if it has no solutions. A system with at least one solution is consistent. A system with a unique solution is said to be independent while a system having infinitely many solutions is said to be dependent.

Example 6.2.1. Solve the system. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 3x-y+z=3 \\ 2x-4y+3z=16 \\ x-y+z=5 \end{cases}$$

Solution. We begin by labeling the equations:

$$\begin{aligned} 3x-y+z&=3 \quad (1) \\ 2x-4y+3z&=16 \quad (2) \\ x-y+z&=5 \quad (3) \end{aligned}$$

We choose z as the variable [2] to eliminate, and equations (1) and (3) as the pair [3] we will start with. After multiplying equation (3) by -1, we add the resulting equations.

$$\begin{array}{ll} 3x-y+z=3 & (1) \\ \underline{-x+y-z=-5} & \text{(3) multiplied by } -1 \\ 2x \qquad\quad =-2 & \end{array}$$

Now we choose equations [4] (1) and (2), multiplying equation (1) by -3 in an attempt to eliminate z.

$$\begin{array}{ll} -9x+3y-3z=-9 & \text{(1) multiplied by } -3 \\ \underline{\;\;2x-4y+3z=16} & (2) \\ -7x\;-y \qquad\;\; =7 & \end{array}$$

We solve the resulting system for x and y.

$$\begin{cases} 2x=-2 \\ -7x-y=7 \end{cases}$$

The first equation, $2x=-2$, simplifies to $x=-1$. Substituting $x=-1$ in the second equation results in $y=0$. We back-substitute these values into equation (1) to find the value of z.

$$\begin{aligned} 3x-y+z&=3 \\ 3(-1)-(0)+z&=3 \\ -3+z&=3 \end{aligned}$$

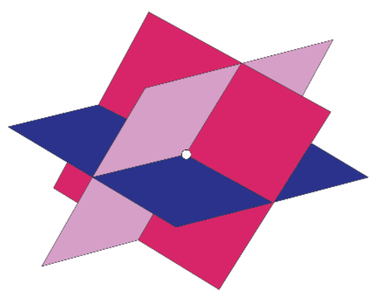
[2] We could just as easily choose *x* or *y*.
[3] Any pair will work here – simply a matter of choice.
[4] Any pair is okay as long as it is different from the first pair we used.

Hence, $z=6$. We leave it to the reader to check that substituting the respective values for x, y, and z in the original system results in three identities. This system is consistent since it has the solution $x=-1$, $y=0$, and $z=6$. It is an independent system since the solution is unique.

□

It is desirable to write the solution to the system in **Example 6.2.1** by extending the usual ordered pair (x,y) notation to 'ordered triple' (x,y,z) and list the solution as $(-1,0,6)$. The question quickly becomes what does an ordered triple like $(-1,0,6)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space. Geometrically, the ordered triple $(-1,0,6)$ is the intersection, or common point, of the three planes represented by the three equations. If you imagine three sheets of paper, each representing a portion of these planes, you will start to see the complexities involved in how three such planes intersect. Below is a sketch of three planes intersecting in a single point.

Figure 6.2. 1



Example 6.2.2. Solve the system. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 2x+3y-z=1 \\ 10x-z=2 \\ 4x-9y+2z=5 \end{cases}$$

Solution. We label the equations.

$$\begin{aligned} 2x+3y-z&=1 \quad (1) \\ 10x-z&=2 \quad (2) \\ 4x-9y+2z&=5 \quad (3) \end{aligned}$$

Since there is no y-term in equation (2), we choose y as the variable to be eliminated. We need another equation with no y-term, so we work with equations (1) and (3); multiply equation (1) by 3 and add it to equation (3).

$$\begin{array}{rl} 6x+9y-3z=3 & \text{(1) multiplied by 3} \\ \underline{4x-9y+2z=5} & \text{(3)} \\ 10x \qquad -z=8 & \end{array}$$

We have the resulting system

$$\begin{cases} 10x-z=2 \\ 10x-z=8 \end{cases}$$

To solve this system, we eliminate x by multiplying the second equation by -1 and adding it to the first equation.

$$\begin{array}{rl} 10x-z=2 & \\ \underline{-10x+z=-8} & \text{after multiplying } 10x-z=8 \text{ by } -1 \\ 0=-6 & \end{array}$$

By eliminating x, we have also eliminated z, with the resulting equation $0=-6$. This is a contradiction so the system has no solution. Thus, we have an inconsistent system.

□

There are three different geometric possibilities for systems of three linear equations that have no solutions. As seen in the first figure, below, the planes may intersect with each other, but not at a common point. In the second figure, two of the planes are parallel and intersect with the third plane, and the three planes do not have any points in common. The third figure illustrates three parallel planes that do not intersect.

Figure 6.2. 2

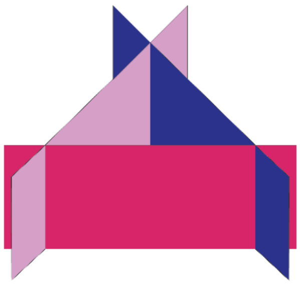


Figure 6.2. 3

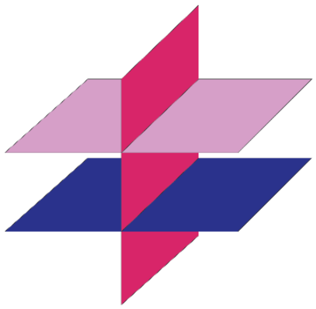
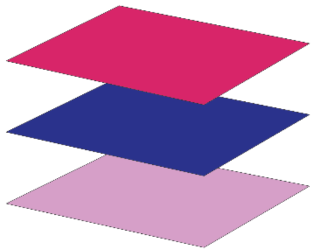


Figure 6.2. 4



Example 6.2.3. Solve the system. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 2x+3y-6z=1 \\ -4x-6y+12z=-2 \\ x+2y+5z=10 \end{cases}$$

Solution. Once again, we start by numbering the equations.

$$\begin{aligned} 2x+3y-6z&=1 \quad (1)\\ -4x-6y+12z&=-2 \quad (2)\\ x+2y+5z&=10 \quad (3) \end{aligned}$$

It looks like x is the easiest variable to eliminate. We begin with equations (1) and (2); multiply equation (1) by 2 and add it to equation (2).

$$\begin{array}{rl} 4x+6y-12z=2 & \text{(1) multiplied by 2} \\ \underline{-4x-6y+12z=-2} & \text{(2)} \\ 0=0 & \end{array}$$

We can also try equations (2) and (3); multiply equation (3) by 4 and add it to equation (2).

$$\begin{array}{rl} -4x-6y+12z=-2 & \text{(2)} \\ \underline{4x+8y+20z=40} & \text{(3) multiplied by 4} \\ 2y\ +32z=38 & \end{array}$$

These two eliminations result in the following system of equations.

$$\begin{cases} 0=0 \\ 2y+32z=38 \end{cases}$$

The first equation, $0=0$, is always true. So, the new system has infinitely many solutions. We **parametrize** the solution set with a parameter t by letting $z=t$. Substituting $z=t$ in the second equation, we have

$$\begin{aligned} 2y+32t&=38\\ y+16t&=19\\ y&=-16t+19 \end{aligned}$$

Substituting $y=-16t+19$ and $z=t$ in the first equation, $2x+3y-6z=1$, gives us

$$\begin{aligned} 2x+3(-16t+19)-6t&=1\\ 2x-48t+57-6t&=1\\ 2x&=54t-56\\ x&=27t-28 \end{aligned}$$

Our solution is the collection of all ordered triples $(27t-28,-16t+19,\ t)$ for all real numbers t. Thus, our system is a consistent dependent system.

□

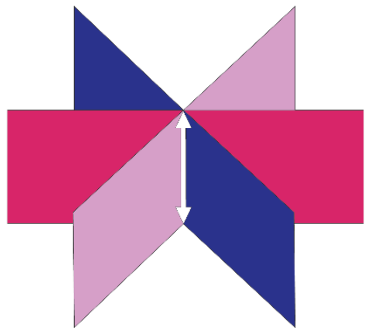
A few notes are in order:

1. We could use any letter for the parameter in place of t, for example s, or just z itself.
2. We could use $y=t$ for the parametrization. In this case, the ordered triple solution would be $\left(-\frac{27}{16}t+\frac{65}{16},\ t,\ -\frac{1}{16}t+\frac{19}{16}\right)$. Of course, the choice of $z=t$ avoided fractions in the solution. We

could have also used $x = t$ for the parametrization, if we had chosen the variable y or z to be eliminated.

3. Geometrically, the solution set is the line of intersection of the three planes. This is illustrated in the following figure.

Figure 6.2. 5



Applications of Linear Equations in Three Variables

Example 6.2.4. John received an inheritance of \$12,000 that he divided into three parts and invested in three ways: a money-market fund paying 3% annual interest, municipal bonds paying 4% annual interest, and mutual funds paying 7% annual interest. John invested \$4,000 more in municipal funds than in municipal bonds. He earned \$670 in interest the first year. How much did John invest in each type of fund?

Solution. To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

Our first equation indicates that the sum of the three investments is \$12,000: $x + y + z = 12000$. We use the information that John invested \$4,000 more in mutual funds than he invested in municipal bonds for a second equation: $z = y + 4000$. The third equation, $0.03x + 0.04y + 0.07z = 670$, shows that the total amount earned from the three funds equals \$670. We write these three equations as a system.

$$\begin{cases} x + y + z = 12000 \\ z = y + 4000 \\ 0.03x + 0.04y + 0.07z = 670 \end{cases}$$

After reordering the variables in the second equation and multiplying the third equation by 100, to simplify calculations, we assign numbers to the equations.

$$\begin{aligned} x+y+z&=12000 \quad (1)\\ -y+z&=4000 \quad (2)\\ 3x+4y+7z&=67000 \quad (3) \end{aligned}$$

Since there is no *x*-term in equation (2), we choose x as the target variable to eliminate first, and proceed with eliminating x from equations (1) and (3).

$$\begin{aligned} -3x-3y-3z&=-36000 \quad \text{(1) multiplied by } -3\\ \underline{3x+4y+7z}&=\underline{\;\;67000} \quad (3)\\ y+4z&=\;\;31000 \end{aligned}$$

We have the resulting system

$$\begin{cases} -y+z=4000 & \text{equation (2)}\\ y+4z=31000 \end{cases}$$

Now we eliminate y from this 2×2 system.

$$\begin{aligned} -y+\;z&=\;\;4000\\ \underline{y+4z}&=\underline{31000}\\ 5z&=35000 \end{aligned}$$

So, $z=7000$ and we find y by substituting into equation (2), $-y+z=4000$.

$$\begin{aligned} -y+(7000)&=4000\\ -y&=-3000\\ y&=3000 \end{aligned}$$

After substituting $y=3000$ and $z=7000$ into equation (1), $x+y+z=12000$, we have

$$\begin{aligned} x+(3000)+(7000)&=12000\\ x&=2000 \end{aligned}$$

To summarize, John invested \$2,000 in a money-market fund, \$3,000 in municipal bonds and \$7,000 in mutual funds.

□

Example 6.2.5. Find the quadratic function passing through the points $(1,6)$, $(-1,10)$, and $(2,13)$.

Solution. Recall that a quadratic function has the form $f(x)=ax^2+bx+c$ where $a\neq 0$. Our goal is to find a, b, and c so that the three given points are on the graph of f. If $(1,6)$ is on the graph, then $f(1)=6$ or $a(1)^2+b(1)+c=6 \Rightarrow a+b+c=6$. Since the point $(-1,10)$ is also on the graph of f, then $f(-1)=10$ which gives us the equation $a-b+c=10$. Lastly, the point $(2,13)$ being on the graph of f gives us $4a+2b+c=13$. The following system, with equations numbered, is the result.

$$\begin{cases} a+b+c=6 & (1) \\ a-b+c=10 & (2) \\ 4a+2b+c=13 & (3) \end{cases}$$

We make b the target variable to eliminate first, and start with equations (1) and (2).

$$\begin{array}{rl} a+b+\ \ c=\ \ 6 & (1) \\ \underline{a-b+\ \ c=10} & (2) \\ 2a\qquad +2c=16 & \end{array}$$

We can eliminate b from equations (2) and (3) by multiplying equation (2) by 2 and adding it to equation (3).

$$\begin{array}{rl} 2a-2b+2c=20 & \text{(2) multiplied by 2} \\ \underline{4a+2b+\ \ c=13} & (3) \\ 6a\qquad +3c=33 & \end{array}$$

The resulting 2×2 system is

$$\begin{cases} 2a+2c=16 \\ 6a+3c=33 \end{cases}$$

We multiply the first equation by −3 and add the result to the second equation to eliminate a.

$$\begin{array}{rl} -6a-6c=-48 & \text{after multiplying } 2a+2c=16 \text{ by } -3 \\ \underline{6a+3c=\ \ 33} & \\ -3c=-15 & \end{array}$$

We find $c=5$ and substitute into the equation $6a+3c=33$ to get $a=3$. Then, substituting both $a=3$ and $c=5$ into equation (1), $a+b+c=6$, we find $b=-2$. We plug these three values into $f(x)=ax^2+bx+c$ and have the quadratic function $f(x)=3x^2-2x+5$ that passes through the points $(1,6)$, $(-1,10)$ and $(2,13)$. We leave it to the reader to check this solution.

□

6.2 Exercises

1. Can a linear system of three equations have exactly two solutions? Explain why or why not.
2. What is the geometric interpretation of a system of linear equations in three variables that is independent? How many solutions are there?

In Exercises 3 – 20, solve the system, if possible. State any solutions and classify each system as consistent independent, consistent dependent, or inconsistent.

3. $\begin{cases} x+y+z=3 \\ 2x-y+z=0 \\ -3x+5y+7z=7 \end{cases}$

4. $\begin{cases} 4x-y+z=5 \\ 2y+6z=30 \\ x+z=5 \end{cases}$

5. $\begin{cases} 4x-y+z=5 \\ 2y+6z=30 \\ x+z=6 \end{cases}$

6. $\begin{cases} x+y+z=-17 \\ y-3z=0 \end{cases}$

7. $\begin{cases} x+2y+z=7 \\ -y+3z=9 \end{cases}$

8. $\begin{cases} x-2y+3z=7 \\ -3x+y+2z=-5 \\ 2x+2y+z=3 \end{cases}$

9. $\begin{cases} 3x-2y+z=-5 \\ x+3y-z=12 \\ x+y+2z=0 \end{cases}$

10. $\begin{cases} 2x-y+z=-1 \\ 4x+3y+5z=1 \\ 5y+3z=4 \end{cases}$

11. $\begin{cases} x-y+z=-4 \\ -3x+2y+4z=-5 \\ x-5y+2z=-18 \end{cases}$

12. $\begin{cases} 2x-4y+z=-7 \\ x-2y+2z=-2 \\ -x+4y-2z=3 \end{cases}$

13. $\begin{cases} 2x-y+z=1 \\ 2x+2y-z=1 \\ 3x+6y+4z=9 \end{cases}$

14. $\begin{cases} x-3y-4z=3 \\ 3x+4y-z=13 \\ 2x-19y-19z=2 \end{cases}$

15. $\begin{cases} x+y+z=4 \\ 2x-4y-z=-1 \\ x-y=2 \end{cases}$

16. $\begin{cases} x-y+z=8 \\ 3x+3y-9z=-6 \\ 7x-2y+5z=39 \end{cases}$

17. $\begin{cases} 2x-3y+z=-1 \\ 4x-4y+4z=-13 \\ 6x-5y+7z=-25 \end{cases}$

18. $\begin{cases} 2x_1+x_2-12x_3-x_4=16 \\ -x_1+x_2+12x_3-4x_4=-5 \\ 3x_1+2x_2-16x_3-3x_4=25 \\ x_1+2x_2-5x_4=11 \end{cases}$

19. $\begin{cases} x_1-x_3=-2 \\ 2x_2-x_4=0 \\ x_1-2x_2+x_3=0 \\ -x_3+x_4=1 \end{cases}$

20. $\begin{cases} x_1-x_2-5x_3+3x_4=-1 \\ x_1+x_2+5x_3-3x_4=0 \\ x_2+5x_3-3x_4=1 \\ x_1-2x_2-10x_3+6x_4=-1 \end{cases}$

21. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

22. You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money into the account that pays 4% compared to 3%, how much did you invest in each account?

23. Find the quadratic function passing through the points $(-1,-4)$, $(1,6)$, and $(3,0)$.

24. Find the quadratic function passing through the points $(1,-1)$, $(3,-1)$, and $(-2,14)$.

25. Find the equation in standard form of the circle passing through the points $(-2,5)$, $(5,4)$, and $(1,-4)$.

26. The top three countries in oil consumption in a certain year are as follows: United States, Japan, and China. In millions of barrels per day, the top three countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?

27. The top three countries in oil production in the same year are Saudi Arabia, United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?

28. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. These top three countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percentage of United States oil imports were from these three countries?

29. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?

30. At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

31. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each type of animal are at the shelter?

32. Your roommate, Prasantha, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate's groceries were $2.10 more than your groceries. How much was each of your shares of the groceries?

33. Three co-workers have jobs as warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more that the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

34. At a carnival, $2,941.25 in receipts was taken by the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

35. A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for students, $22.50 for children, and $29 for adults. If the band sold twice as many adult as child tickets, how many of each type were sold?

36. In a bag, a child has 325 coins worth $19.50. There are three types of coins: pennies, nickels, and dimes. If the bag contains the same number of nickels as dimes, how many of each type of coin is in the bag?

6.3 Systems of Linear Equations: Augmented Matrices

Learning Objectives

- Represent a system of linear equations as an augmented matrix.
- Perform row operations on a matrix.
- Convert an augmented matrix to row echelon form.
- Convert an augmented matrix to reduced row echelon form.
- Use matrix row operations to solve systems of linear equations.

In **Section 6.1** and **Section 6.2**, we solved systems of linear equations using the substitution method, the elimination method, and combinations of these two methods. The goal of these methods was to rewrite the original equations in a way that would allow us to determine solution values. In this section, our goal is to rewrite a system of equations in a format similar to the following.

$$\begin{cases} x-3y+2z=1 \\ \quad\;\; y-2z=4 \\ \qquad\quad\; z=-1 \end{cases}$$

Here, clearly, $z=-1$, and we substitute $z=-1$ in the second equation to obtain $y-2(-1)=4$, resulting in the value $y=2$. Then we substitute $y=2$ and $z=-1$ in the first equation to get $x-3(2)+2(-1)=1$, from which $x=9$. The solution is $(9,2,-1)$.

We note that the reason it was so easy to solve this system is that the third equation is solved for z and the second involves only y and z. Additionally, the coefficient of y is 1 in the second equation and the coefficient of x is 1 in the first equation. It will be our goal in this section to rewrite systems in this form, referred to as **upper triangular form**.[5]

Row Operations

To write a system of linear equations in triangular form, we may apply the following maneuvers to achieve an equivalent system.[6]

[5] We will sometimes refer to this simply as **triangular form**.
[6] That is, a system with the same solution set.

Theorem 6.1. Given a system of equations, the following operations will result in an equivalent system of equations.

- Interchange the positions of any two equations.
- Replace an equation with a nonzero multiple of itself.
- Replace an equation with itself plus a multiple of another equation.

We have seen instances of the second and third operations in examples from the previous two sections. The first operation, while it obviously admits an equivalent system, seems silly. Our perspective will change as we begin solving systems using this methodology. The first example is a system we originally solved in **Section 6.2**:

$$\begin{cases} 3x-y+z=3 \\ 2x-4y+3z=16 \\ x-y+z=5 \end{cases}$$

As we attempt to write this system in triangular form, we will mimic our moves in a matrix representation of the system. A **matrix** is simply a rectangular array of real numbers, enclosed with square brackets, '[' and ']'. To represent a system as a matrix, we include only coefficients and constants. The rows represent equations and columns correspond to the coefficients of specific variables. Since each column corresponds to a specific variable (generally column 1 for x, column 2 for y, and column 3 for z), before representing a system of equations in matrix format, all equations must be written in the form $ax+by+cz=d$. Noting that this is indeed the case in the above system, we write it in matrix format.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$$

The vertical line is not necessary, but its presence tells us that this is an **augmented matrix**; the column containing the constants is 'appended' to the matrix containing the coefficients.

Example 6.3.1. Put the following system of linear equations into triangular form and then solve the system.

$$\begin{cases} 3x-y+z=3 \\ 2x-4y+3z=16 \\ x-y+z=5 \end{cases}$$

Solution. To introduce matrix solution methods, we will solve this problem using both the system of equations and its augmented matrix representation. We name each equation as follows: E_1 refers to the top equation, E_2 refers to the middle equation, and E_3 refers to the bottom equation. In the matrix, we use R_1 for 'row one', R_2 for 'row two', and R_3 for 'row three'; or top row, middle row, bottom row, respectively. Our first step is to get an x having a coefficient of 1 in the first equation, so we begin by interchanging E_1 with E_3.

$$\begin{cases} 3x-y+z=3 \\ 2x-4y+3z=16 \\ x-y+z=5 \end{cases} \xrightarrow{\text{Switch } E_1 \text{ and } E_3} \begin{cases} x-y+z=5 \\ 2x-4y+3z=16 \\ 3x-y+z=3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{\text{Switch } R_1 \text{ and } R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right]$$

Now we need to eliminate x's from the second and third equations , so we replace each with a sum of that equation and a multiple of the first equation. To eliminate x from E_2, we multiply E_1 by -2 then add; to eliminate x from E_3, we multiply E_1 by -3 then add.

$$\begin{cases} x-y+z=5 \\ 2x-4y+3z=16 \\ 3x-y+z=3 \end{cases} \xrightarrow[\text{Replace } E_3 \text{ with } -3E_1+E_3]{\text{Replace } E_2 \text{ with } -2E_1+E_2} \begin{cases} x-y+z=5 \\ -2y+z=6 \\ 2y-2z=-12 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right] \xrightarrow[\text{Replace } R_3 \text{ with } -3R_1+R_3]{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & -2 & 1 & 6 \\ 0 & 2 & -2 & -12 \end{array}\right]$$

Next, we change the coefficient of y in the second row to 1. This is accomplished through multiplying the second row by $-\frac{1}{2}$.

$\begin{cases} x-y+z=5 \\ -2y+z=6 \\ 2y-2z=-12 \end{cases}$	$\xrightarrow{\text{Replace } E_2 \text{ with } -\frac{1}{2}E_2}$	$\begin{cases} x-y+z=5 \\ y-\frac{1}{2}z=-3 \\ 2y-2z=-12 \end{cases}$
$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 5 \\ 0 & -2 & 1 & 6 \\ 0 & 2 & -2 & -12 \end{array}\right]$	$\xrightarrow{\text{Replace } R_2 \text{ with } -\frac{1}{2}R_2}$	$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 2 & -2 & -12 \end{array}\right]$

We move on to eliminating y in the third equation; we multiply the second equation by -2, then add.

$\begin{cases} x-y+z=5 \\ y-\frac{1}{2}z=-3 \\ 2y-2z=-12 \end{cases}$	$\xrightarrow{\text{Replace } E_3 \text{ with } -2E_2+E_3}$	$\begin{cases} x-y+z=5 \\ y-\frac{1}{2}z=-3 \\ -z=-6 \end{cases}$
$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 2 & -2 & -12 \end{array}\right]$	$\xrightarrow{\text{Replace } R_3 \text{ with } -2R_2+R_3}$	$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & -1 & -6 \end{array}\right]$

Finally, we change the coefficient of z to 1 by multiplying the third equation by -1.

$\begin{cases} x-y+z=5 \\ y-\frac{1}{2}z=-3 \\ -z=-6 \end{cases}$	$\xrightarrow{\text{Replace } E_3 \text{ with } -1\cdot E_3}$	$\begin{cases} x-y+z=5 \\ y-\frac{1}{2}z=-3 \\ z=6 \end{cases}$
$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & -1 & -6 \end{array}\right]$	$\xrightarrow{\text{Replace } R_3 \text{ with } -1\cdot R_3}$	$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & 1 & 6 \end{array}\right]$

The system is in triangular form. We see $z=6$, which we plug into the second equation to get $y-\frac{1}{2}(6)=-3$, so that $y=0$. Then, plugging in values for y and z, the first equation becomes

$x-0+6=5$, or $x=-1$. According to **Theorem 6.1**, since this system in triangular form has the solution $(-1,0,6)$, so does the original system.

□

Row Echelon Form of a Matrix

The matrix equivalent of triangular form is **row echelon form**, as seen in the previous example.

Definition 6.3. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1. This is referred to as a 'leading 1'.
2. The leading 1 of a given row is to the right of the leading 1 in any row above it.
3. A row of all zeros is below all rows with any nonzero entries.

The following matrices are in row echelon form. Note that each matrix fulfills the requirements of **Definition 6.3**. Additionally, the third matrix is in what we refer to as reduced row echelon form, a classification we will discuss later in this section.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 4 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Below are examples of matrices that are not in row echelon form. Can you see where they fail to pass the criteria of **Definition 6.3**?

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 4 & 5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 6 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \end{array}\right]$$

The strategy used to obtain the row-echelon form of a matrix, as demonstrated in **Example 6.3.1**, is known as Gaussian Elimination, named after the prolific German mathematician Carl Friedrich Gauss. It provides us with a systematic way to solve systems of linear equations, though not always the shortest way. To solve a system of linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to get the matrix into row-echelon form. We then decode the matrix and solve the system using back-substitution.

The connection between equations in a system and rows in a matrix allows us to move easily from the maneuvers allowed in **Theorem 6.1** to the row operations allowed in the following theorem.

Theorem 6.2. Row Operations: Given an augmented matrix representing a system of linear equations, the following row operations produce an augmented matrix that corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[7]
- Replace a row with itself plus a multiple of another row.[8]

Example 6.3.2. Solve the system of equations, if possible, by writing the system as an augmented matrix and putting the augmented matrix in row echelon form.

$$\begin{cases} 2x+3y=6 \\ x-y=\dfrac{1}{2} \end{cases}$$

Solution. We begin by encoding the system into an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right]$$

We now attempt to get the matrix into row echelon form. First of all, we get a 1 in the first row, first column, by interchanging the first and second rows.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{\text{Switch } R_1 \text{ and } R_2} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right]$$

Next, dropping down to the second row of the first column, we obtain a 0 by multiplying the first row by -2, then adding the result to the second row.

$$\left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right]$$

To get a 1 in the second column of the second row, we multiply the second row by $\frac{1}{5}$.

$$\left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } \frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array}\right]$$

Now that our matrix is in row echelon form, we decode by writing the corresponding system of equations.

---

[7] That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[8] Where we add entries in corresponding columns.

$$\begin{cases} x - y = \dfrac{1}{2} \\ y = 1 \end{cases}$$

We back-substitute $y = 1$ into the equation $x - y = \frac{1}{2}$ to get $x - 1 = \frac{1}{2}$, or $x = \frac{3}{2}$. The solution is $\left(\frac{3}{2}, 1\right)$.

□

Now that we've worked through a couple of examples, you may notice that we have followed the same order in getting the 1's and 0's in the required positions to achieve row echelon form. While there is more than one way to reach row echelon form, the following order of operations will prevent 'undoing' a correct placement of 1's or 0's that has already been made.

Figure 6.3. 1

$$\left[\begin{array}{cc|c} \overset{\text{1st}}{1} & \# & \# \\ \overset{\text{2nd}}{0} & \overset{\text{3rd}}{1} & \# \end{array}\right]$$

Figure 6.3. 2

$$\left[\begin{array}{ccc|c} \overset{\text{1st}}{1} & \# & \# & \# \\ \overset{\text{2nd}}{0} & \overset{\text{4th}}{1} & \# & \# \\ \overset{\text{3rd}}{0} & \overset{\text{5th}}{0} & \overset{\text{6th}}{1} & \# \end{array}\right]$$

To the left is the order for a matrix with two rows and to the right is the order for a matrix with three rows. Keep in mind that rows of zeros may appear through this process and should be moved to the bottom.

Note that it is not necessary to get a matrix fully into row echelon form in order to solve the system of equations. At any point along the way, the matrix can be decoded and the resulting system of equations solved.

Example 6.3.3. Solve the system of equations, by representing the system as an augmented matrix and converting the augmented matrix to row echelon form.

$$\begin{cases} -x - 2y + z = -1 \\ 2x + 3y = 2 \\ y - 2z = 0 \end{cases}$$

Solution. Encoding the system as an augmented matrix, we have

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

To obtain a leading coefficient of 1 in the first row, we multiply by -1.

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -1\cdot R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

Now, we want zeros down the first column below row 1. For the second row, we multiply row one by -2 and add the result to row 2. We already have a zero in row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

The next step requires obtaining a 1 in the second row, second column. We swap row 2 and row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Switch } R_2 \text{ and } R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

We need a 0 below the 1 in row 2, so we multiply row 2 by 1 and add it to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } 1\cdot R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is now in row echelon form. We decode it to arrive at the following system of equations.

$$\begin{cases} x+2y-z=1 \\ \quad y-2z=0 \\ \qquad 0=0 \end{cases}$$

The equation $0=0$ is always true. So, the new system has infinitely many solutions and, therefore, our original system is a consistent dependent system. We select t as the parameter and set $z=t$. Substituting $z=t$ in the second equation, we have

$$\begin{aligned} y-2t&=0 \\ y&=2t \end{aligned}$$

Substituting $y=2t$ and $z=t$ in the first equation, $x+2y-z=1$, gives us

$$\begin{aligned} x+2(2t)-t&=1 \\ x+3t&=1 \\ x&=-3t+1 \end{aligned}$$

Our solution is the collection of ordered triples $(-3t+1, 2t, t)$ for all real numbers t.

□

Reduced Row Echelon Form of a Matrix

To get a matrix into row echelon form, we used row operations to obtain 0's beneath each leading 1. If we also require that 0's are the only numbers above a leading 1, we have what is known as the **reduced row echelon form** of the matrix.

Definition 6.4. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.
2. The leading 1's are the only nonzero entries in their respective columns.

The following matrices are in row echelon form. Use **Definition 6.4** to determine which of these matrices is/are also in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

The only matrix that is not in reduced row echelon form is the second matrix. The second column of this matrix has a leading 1 in the second row, and a nonzero entry of 2 in the first row. Take some time to verify that the first, third, and fourth matrices do indeed meet the criteria for being in reduced row echelon form.

In **Example 6.3.3**, we applied row operations to get the matrix into the following row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To go a step further and put the matrix in reduced row echelon form, we must have a 0 in place of the 2 in row 1. We achieve this by multiplying row 2 by -2, then adding the result to row 1.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note that taking the extra step to write this matrix in reduced row echelon form would save a few steps in the decoding process.[9] Following is a **suggested** order for using row operations to transform a matrix

[9] Try it!

into reduced row echelon form. The first shows the order for any matrix with two rows and the second for a matrix with three rows.

Figure 6.3. 3

$$\left[\begin{array}{cc|c} \overset{\text{1st}}{1} & \overset{\text{4th}}{0} & \# \\ \overset{\text{2nd}}{0} & \overset{\text{3rd}}{1} & \# \end{array}\right]$$

Figure 6.3. 4

$$\left[\begin{array}{ccc|c} \overset{\text{1st}}{1} & \overset{\text{8th}}{0} & \overset{\text{9th}}{0} & \# \\ \overset{\text{2nd}}{0} & \overset{\text{4th}}{1} & \overset{\text{7th}}{0} & \# \\ \overset{\text{3rd}}{0} & \overset{\text{5th}}{0} & \overset{\text{6th}}{1} & \# \end{array}\right]$$

Example 6.3.4. Solve the system of equations, by representing the system as an augmented matrix and converting the augmented matrix into reduced row echelon form.

$$\begin{cases} x+y+z=4 \\ 2x-y-2z=-1 \\ x-2y-z=1 \end{cases}$$

Solution. We first write the system as an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & -2 & -1 \\ 1 & -2 & -1 & 1 \end{array}\right]$$

Since there is a leading 1 in the first row, we proceed with obtaining zeros in rows 2 and 3 of the first column.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & -2 & -1 \\ 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow[\text{Replace } R_3 \text{ with } -1\cdot R_1+R_3]{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -4 & -9 \\ 0 & -3 & -2 & -3 \end{array}\right]$$

Now we obtain a leading 1 in row 2 through multiplication by $-\dfrac{1}{3}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -4 & -9 \\ 0 & -3 & -2 & -3 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & -3 & -2 & -3 \end{array}\right]$$

Next, we get a 0 in the third row, second column.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & -3 & -2 & -3 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } 3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

We are almost to row echelon form; just need a leading 1 in row 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 2 & 6 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now that the matrix is in row echelon form, we need to make sure any columns with leading 1's have 0's elsewhere. In row 2, we need a 0 in the position currently occupied by $\frac{4}{3}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -\frac{4}{3}R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

In row 1, we need zeros in both column 2 and column 3. To follow the same pattern, we move from left to right and start with column 2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -1\cdot R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now we take care of column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -1\cdot R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

With the matrix in reduced row echelon form, we decode it to get a system of equations.

$$\begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

To our surprise and delight, when we decoded this matrix, we obtained the solution instantly without having to deal with back-substitution.

□

Before moving on to the next section, we note that it is possible to use matrices in solving systems with any number of variables. Following is an example of a system with the four variables A, B, C, and D.

Example 6.3.5. Solve the system of equations by representing the system as an augmented matrix and converting the augmented matrix into reduced row echelon form.

$$\begin{cases} A = 1 \\ B = 0 \\ 3A + C = 5 \\ 3B + D = -1 \end{cases}$$

Solution. We begin by representing the system as an augmented matrix. Note that we have coefficients of A, B, C, and D in columns 1, 2, 3, and 4, respectively.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 5 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right]$$

The first entry we will work on is the 3 in the third row, first column. We proceed with obtaining a zero in that position.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 5 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } -3R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right]$$

Next, we get a 0 in row 4, the second column.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R_4 \text{ with } -3R_2 + R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Having the matrix in reduced row echelon form, we can decode it to find the solutions.

$$\begin{cases} A = 1 \\ B = 0 \\ C = 2 \\ D = -1 \end{cases}$$

□

We will refer back to the solution in **Example 6.3.5** when we get to partial fraction decomposition in **Section 6.7**. For now, we move on to matrix arithmetic in **Section 6.4**.

6.3 Exercises

1. What are two different row operations that can be used to obtain a leading 1 in the first row of the matrix $\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 1 & -2 & 6 \end{array}\right]$?

2. What does a row of 0's in an augmented matrix tell you about the system of equations it is representing?

In Exercises 3 – 6, state whether the given matrix is in reduced row echelon form, row echelon form only, or neither of these.

3. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

In Exercises 7 – 10, the matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

7. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

10. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 1 \end{array}\right]$

In Exercises 11 – 26, solve the systems of linear equations using elementary row operations on an augmented matrix.

11. $\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$

12. $\begin{cases} 2x - 3y = -9 \\ 5x + 4y = 58 \end{cases}$

13. $\begin{cases} 2x + 3y = 12 \\ 4x + y = 14 \end{cases}$

14. $\begin{cases} 3x + 4y = 12 \\ -6x - 8y = -24 \end{cases}$

15. $\begin{cases} x+y+z=3 \\ 2x-y+z=0 \\ -3x+5y+7z=7 \end{cases}$

16. $\begin{cases} 4x-y+z=5 \\ 2x+6z=30 \\ x+z=5 \end{cases}$

17. $\begin{cases} x-2y+3z=7 \\ -3x+y+2z=-5 \\ 2x+2y+z=3 \end{cases}$

18. $\begin{cases} 3x-2y+z=-5 \\ x+3y-z=12 \\ x+y+2z=0 \end{cases}$

19. $\begin{cases} 2x-y+z=-1 \\ 4x+3y+5z=1 \\ 5y+3z=4 \end{cases}$

20. $\begin{cases} x-y+z=-4 \\ -3x+2y+4z=-5 \\ x-5y+2z=-18 \end{cases}$

21. $\begin{cases} 2x-4y+z=-7 \\ x-2y+2z=-2 \\ -x+4y-2z=3 \end{cases}$

22. $\begin{cases} 2x-y+z=1 \\ 2x+2y-z=1 \\ 3x+6y+4z=9 \end{cases}$

23. $\begin{cases} x-3y-4z=3 \\ 3x+4y-z=13 \\ 2x-19y-19z=2 \end{cases}$

24. $\begin{cases} x+y+z=4 \\ 2x-4y-z=-1 \\ x-y=2 \end{cases}$

25. $\begin{cases} x-y+z=8 \\ 2x+3y-9z=-6 \\ 7x-2y+5z=39 \end{cases}$

26. $\begin{cases} 2x-3y+z=-1 \\ 4x-4y+4z=-13 \\ 6x-5y+7z=-25 \end{cases}$

27. At the local buffet, 22 diners (5 of whom were children) feasted for \$162.25, before taxes. If the kids buffet is \$4.50, the basic buffet is \$7.50, and the deluxe buffet (with crab legs) is \$9.25, how many diners chose the deluxe buffet?

28. Jianming wants to make a party mix consisting of almonds (which cost \$7 per pound), cashews (which cost \$5 per pound), and peanuts (which cost \$2 per pound). If he wants to make a 10 pound mix with a budget of \$35, what are the possible combinations of almonds, cashews, and peanuts?

29. Find the quadratic function passing through the points $(-2,1)$, $(1,4)$, and $(3,-2)$.

30. Find two different row echelon forms for the matrix $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 12 & 8 \end{array}\right]$.

6.4 Matrix Arithmetic

Learning Objectives

- Find the sum and difference of two matrices.
- Find the scalar multiple of a matrix.
- Find the product of two matrices.
- Apply properties of matrices.

In **Section 6.3**, we used a special class of matrices, the augmented matrices, to assist us in solving systems of linear equations. In this section, we study matrices as mathematical objects of their own accord, temporarily detached from the systems of linear equations. Recall that a **matrix** is a rectangular array of real numbers, enclosed with square brackets, '[' and ']', such as the following.

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$$

The **size**, sometimes called the **dimension**, of the matrix shown above is 2×3, read as 'two by three', because it has 2 rows and 3 columns. In general, we say a matrix with m rows and n columns is of size 'm by n', written as $m\times n$. Before moving on to properties and operations, we note that the individual numbers in a matrix are called its **entries** and that matrices are usually denoted by uppercase letters (A, B, C, etc.). We begin with a definition of what it means for two matrices to be equal.

Definition 6.5. Matrix Equality: Two matrices are said to be **equal** if they are the same size and their corresponding entries are equal.

Essentially, two matrices are equal if they are the same size and they have the same numbers in the same spots. Matrix equality is used to find the value of the variables in the following matrices:

$$\begin{bmatrix} x+2 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & y+1 \end{bmatrix}$$

We must have $x+2=3$ and $y+1=7$, so $x=1$ and $y=6$. Now that we have an understanding of what it means for two matrices to be equal, we begin defining arithmetic operations on matrices, starting with addition.

Matrix Addition

Definition 6.6. Matrix Addition: Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices.

Example 6.4.1. Find the sum, $A+B$, for matrices A and B, shown below.

$$A=\begin{bmatrix}2 & 3\\ 4 & -1\\ 0 & -7\end{bmatrix},\ B=\begin{bmatrix}-1 & 4\\ -5 & -3\\ 8 & 1\end{bmatrix}$$

Solution.

$$A+B=\begin{bmatrix}2 & 3\\ 4 & -1\\ 0 & -7\end{bmatrix}+\begin{bmatrix}-1 & 4\\ -5 & -3\\ 8 & 1\end{bmatrix}=\begin{bmatrix}2+(-1) & 3+4\\ 4+(-5) & (-1)+(-3)\\ 0+8 & (-7)+1\end{bmatrix}=\begin{bmatrix}1 & 7\\ -1 & -4\\ 8 & -6\end{bmatrix}$$

□

It is worth the reader's time to think what would have happened had we reversed the order of the summands above. As we would expect, we arrive at the same answer. In general, $A+B=B+A$ for matrices A and B. This is the **commutative property** of matrix addition. Since addition of matrices is done entry by entry, and each entry is a real number, the commutative property of real number addition carries over to matrices.

The **associative property** of matrix addition also holds, again inherited from the associative law of real number addition. Specifically, for matrices A, B and C of the same size, $(A+B)+C=A+(B+C)$. In other words, when adding more than two matrices, it doesn't matter how they are grouped. This means we can write $A+B+C$ without parentheses since the sum is always the same.

These properties and more are summarized in the following theorem. The matrix referred to as $\mathbf{0}$ is the **zero matrix**, which is a matrix whose entries are all 0. The zero matrix may be any size. For example, $\mathbf{0}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ is the 2×3 zero matrix.

Theorem 6.3. Properties of Matrix Addition. Suppose matrices A, B, C, and the zero matrix $\mathbf{0}$ have the same size.

- **Commutative Property:** $A+B=B+A$
- **Associative Property:** $(A+B)+C=A+(B+C)$
- **Identity Property:** $A+\mathbf{0}=\mathbf{0}+A=A$
- **Inverse Property:** Any matrix has a unique **additive inverse;** for example, the inverse of matrix A is denoted $-A$, where $A+(-A)=(-A)+A=\mathbf{0}$.

The identity property is easily verified by resorting to the definition of matrix addition; just as the number 0 is the additive identity for real numbers, the matrix comprised of all 0's does the same job for matrices.

To establish the inverse property, given a matrix A, we are looking for a matrix $-A$, of the same size, such that all corresponding entries in A and $-A$ add to 0. Thus, to get the additive inverse of the matrix A, we must use the additive inverse of each of its entries to form the matrix we refer to as $-A$. So, if

$A=\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$, then $-A=\begin{bmatrix} -3 & 0 & 1 \\ -2 & 5 & -10 \end{bmatrix}$ since $\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}+\begin{bmatrix} -3 & 0 & 1 \\ -2 & 5 & -10 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Matrix Subtraction

With the concept of additive inverse well in hand, we may now discuss what is meant by subtracting matrices. You may remember from arithmetic that $a-b=a+(-b)$; that is, subtraction is defined as 'adding the (additive) inverse'. We extend this concept to matrices. For two matrices A and B of the same size, we define $A-B=A+(-B)$.

Example 6.4.2. Find $A-B$ for $A=\begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 8 & -1 \\ 5 & 4 \end{bmatrix}$.

Solution.

$$\begin{aligned} A-B &= A+(-B) \\ &= \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} -8 & 1 \\ -5 & -4 \end{bmatrix} \\ &= \begin{bmatrix} (-2)+(-8) & 3+1 \\ 0+(-5) & 1+(-4) \end{bmatrix} \\ &= \begin{bmatrix} -10 & 4 \\ -5 & -3 \end{bmatrix} \end{aligned}$$

□

Scalar Multiplication

Our next task is to define what it means to multiply a matrix by a real number. It is natural to define $3A$ as $A+A+A$ for any matrix A. The matrix $3A$ may be obtained by multiplying each entry of A by 3, leading to the following definition.

Definition 6.7. Scalar[10] Multiplication: We define the product of a real number k and a matrix A, denoted as kA, to be the matrix obtained by multiplying each entry of A by k.

Like matrix addition, scalar multiplication inherits some properties from real number arithmetic. We leave the discovery of these properties to the reader, with the exception of the following additive inverse property.

[10] The word 'scalar' here refers to a real number. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar).

Theorem 6.4. Additive Inverse Property: For all matrices A, $A+(-A)=\mathbf{0}$ where $-A=(-1)A$.

This property is easily verified, and adds clarity to our earlier attempt at finding inverse matrices since the $-A$ from **Theorem 6.4** is indeed the additive inverse of the matrix A.

Example 6.4.3. For matrices A and B, given below, find $2A-3B$.

$$A=\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix},\ B=\begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{aligned}
2A-3B &= 2A+(-3)B \\
&= 2\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix}+(-3)\begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2(-1) & 2\cdot 0 & 2\cdot 3 \\ 2\cdot 2 & 2(-1) & 2\cdot 4 \end{bmatrix}+\begin{bmatrix} (-3)\cdot 2 & (-3)\cdot 1 & (-3)(-3) \\ (-3)\cdot 4 & (-3)\cdot 0 & (-3)\cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 0 & 6 \\ 4 & -2 & 8 \end{bmatrix}+\begin{bmatrix} -6 & -3 & 9 \\ -12 & 0 & -3 \end{bmatrix} \\
&= \begin{bmatrix} -2+(-6) & 0+(-3) & 6+9 \\ 4+(-12) & -2+0 & 8+(-3) \end{bmatrix} \\
&= \begin{bmatrix} -8 & -3 & 15 \\ -8 & -2 & 5 \end{bmatrix}
\end{aligned}$$

□

Note that the above operations can be perfomed a bit more quickly as follows.

$$\begin{aligned}
2A-3B &= \begin{bmatrix} 2(-1)-3\cdot 2 & 2\cdot 0-3\cdot 1 & 2\cdot 3-3(-3) \\ 2\cdot 2-3\cdot 4 & 2(-1)-3\cdot 0 & 2\cdot 4-3\cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} -8 & -3 & 15 \\ -8 & -2 & 5 \end{bmatrix}
\end{aligned}$$

Matrix Multiplication

We now turn our attention to **matrix multiplication** – that is, multiplying a matrix by another matrix. We begin by demonstrating the procedure for finding the product of a row and a column. Consider the two matrices A and B below.

$$A=\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix},\ B=\begin{bmatrix} 3 & 0 & 5 \\ -1 & 2 & 4 \end{bmatrix}$$

Let R_1 denote the first row of A and C_1 denote the first column of B. To find the 'product' of R_1 with C_1, denoted R_1C_1, we begin by finding the product of the first entry in R_1 and the first entry in C_1. To that result, we add the product of the second entry in R_1 and the second entry in C_1. We can visualize this as follows.

$$R_1C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (1)(3)+(2)(-1) = 1$$

To find the product of the second row of A and the third column of B, denoted by R_2C_3, we proceed similarly.

$$R_2C_3 = \begin{bmatrix} -2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = (-2)(5)+(3)(4) = 2$$

In general, multiplication of a single row matrix of size $1\times n$ with a single column matrix of size $n\times 1$ is the sum of products of corresponding entries: $R_1C_1 + R_2C_2 + \cdots + R_nC_n$. Noting that the definition of multiplication of a row by a column works only if the number of entries in the row matches the number of entries in the column, we are now in the position to define matrix multiplication.

Definition 6.8. Matrix Multiplication: Suppose A is an $m\times p$ matrix and B is a $p\times n$ matrix. Let R_i denote the i^{th} row of A and C_j denote the j^{th} column of B. The **product of *A* and *B***, denoted AB, is the matrix defined by

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_n \\ R_2C_1 & R_2C_2 & \cdots & R_2C_n \\ \vdots & \vdots & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_n \end{bmatrix}$$

Note that in light of the discussion of multiplying a row by a column above, the product of an $m\times p$ matrix with a $q\times n$ matrix is defined only when $p=q$, and in this case, the result is an $m\times n$ matrix. It may help to remember these results with the following.

Figure 6.4. 1

$$\underbrace{\begin{bmatrix} \square & \cdots & \square \\ \vdots & \ddots & \vdots \\ \square & \cdots & \square \end{bmatrix}}_{\boldsymbol{m}\times\boldsymbol{p}} \bullet \underbrace{\begin{bmatrix} \square & \cdots & \square \\ \vdots & \ddots & \vdots \\ \square & \cdots & \square \end{bmatrix}}_{\boldsymbol{q}\times\boldsymbol{n}}$$

Must have $\boldsymbol{p} = \boldsymbol{q}$

Dimension of final product: $\boldsymbol{m}\times\boldsymbol{n}$

Returning to our example matrices, shown again below, we see that A is a $\mathbf{2}\times\mathbf{2}$ matrix and B is a $\mathbf{2}\times\mathbf{3}$ matrix. This means that the product matrix AB is defined and will be a $\mathbf{2}\times\mathbf{3}$ matrix.

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 0 & 5 \\ -1 & 2 & 4 \end{bmatrix}$$

Using R_i to denote the i^{th} row of A and C_j to denote the j^{th} column of B, we form AB according to **Definition 6.8**.

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$

We have already determined that $R_1C_1 = 1$ and $R_2C_3 = 2$, as shown in the following diagram.

$$\begin{array}{cccc} & & & B \\ & & & \begin{bmatrix} 3 & 0 & 5 \\ -1 & 2 & 4 \end{bmatrix} \\ & & & \begin{matrix} \downarrow & & \downarrow \end{matrix} \\ A & \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} & \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} & \begin{bmatrix} 1 & \square & \square \\ \square & \square & 2 \end{bmatrix} \end{array}$$

We compute the remaining entries:

$$R_1C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = (1)(0) + (2)(2) = 4$$

$$R_1C_3 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = (1)(5) + (2)(4) = 13$$

$$R_2C_1 = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (-2)(3) + (3)(-1) = -9$$

$$R_2C_2 = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = (-2)(0) + (3)(2) = 6$$

Thus, $AB = \begin{bmatrix} 1 & 4 & 13 \\ -9 & 6 & 2 \end{bmatrix}$. To save a bit of writing, the preceding steps may be combined as follows:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 5 \\ -1 & 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} (1)(3)+(2)(-1) & (1)(0)+(2)(2) & (1)(5)+(2)(4) \\ (-2)(3)+(3)(-1) & (-2)(0)+(3)(2) & (-2)(5)+(3)(4) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & 13 \\ -9 & 6 & 2 \end{bmatrix} \end{aligned}$$

Note that the product BA is not defined, since B is a $\mathbf{2} \times \mathbf{3}$ matrix while A is a $\mathbf{2} \times \mathbf{2}$ matrix; B has more columns than A has rows, and so it is not possible to multiply a row of B by a column of A.

Even when the dimensions of A and B are compatible such that AB and BA are both defined, the products AB and BA are not necessarily equal.[11] In other words, AB may not equal BA. For example, if $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ 3 & -1\end{bmatrix}$, we have

$$AB=\begin{bmatrix}8 & -2\\ 18 & -4\end{bmatrix} \text{ while } BA=\begin{bmatrix}2 & 4\\ 0 & 2\end{bmatrix}$$

Although there is no commutative property of matrix multiplication in general, several other real number properties are inherited by matrix multiplication, as stated in our next theorem.

Theorem 6.5. Properties of Matrix Multiplication Let A, B, and C be matrices such that all of the matrix products below are defined and let r be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$
- **Associative Property with Scalar Multiplication:** $r(AB)=(rA)B=A(rB)$
- **Identity Property:** For a natural number k, the identity matrix I_k is a $k\times k$ (square) matrix containing 1's down the main diagonal (defined below) and 0's elsewhere. If A is an $m\times n$ matrix, then $I_m A = AI_n = A$.
- **Distributive Property of Matrix Multiplication over Matrix Addition:**

$$A(B\pm C)=AB\pm AC \text{ and } (A\pm B)C = AC\pm BC$$

The one property in **Theorem 6.5** that requires further discussion is the identity property. The **main diagonal** that is mentioned refers to positions in the matrix whose row number and column number are the same. As stated, all entries along the main diagonal are 1's while the matrix contains 0's in all other positions. A few examples of identity matrices are presented below.

$$\underset{I_1}{\begin{bmatrix}1\end{bmatrix}} \qquad \underset{I_2}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \qquad \underset{I_3}{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}} \qquad \underset{I_4}{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}$$

A **square matrix** is a matrix that has the same number of rows as columns. An identity matrix must be a square matrix. Note that in order to verify that the identity matrix acts as a multiplicative identity, some

[11] They may not even have the same dimensions. For example, if A is a 2×3 matrix and B is a 3×2 matrix, then AB is defined and is a 2×2 matrix while BA is also defined, but is a 3×3 matrix!

care must be taken depending on the order of the multiplication. For example, take the 2×3 matrix A from earlier.

$$A=\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

In order for the product $I_m A$ to be defined, $m=2$; for AI_n to be defined, $n=3$. We leave it to the reader to verify that $I_2 A = A$ and $AI_3 = A$. In other words,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}=\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

While the proofs of the properties in **Theorem 6.5** are computational in nature, the notation becomes quite involved very quickly, so they are left to a course in Linear Algebra. The following examples provide some practice with matrix multiplication and its properties.

Example 6.4.4. Matrices A, B, C and D are defined as follows:

$$A=\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix},\ B=\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix},\ C=\begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix}, \text{ and } D=\begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Perform the indicated arithmetic operations, if defined, for the given matrices.

(a) $A+D$ (b) $A+B$ (c) BC (d) CB (e) BD (f) DB

Solution.

(a)

$$\begin{aligned} A+D &= \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}+\begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1+0 & 2+(-1) \\ 0+(-2) & -3+3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

(b) $A+B$ is not defined since A is size 2×2 and B is size 2×3. It is only possible to add matrices of the same size.

(c)

$$BC = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (1)(0)+(-1)(-2)+(2)(1) & (1)(1)+(-1)(3)+(2)(0) \\ (0)(0)+(2)(-2)+(1)(1) & (0)(1)+(2)(3)+(1)(0) \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 \\ -3 & 6 \end{bmatrix}$$

(d)

$$CB = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} (0)(1)+(1)(0) & (0)(-1)+(1)(2) & (0)(2)+(1)(1) \\ (-2)(1)+(3)(0) & (-2)(-1)+(3)(2) & (-2)(2)+(3)(1) \\ (1)(1)+(0)(0) & (1)(-1)+(0)(2) & (1)(2)+(0)(1) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 & 1 \\ -2 & 8 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(e) BD is not defined since B is size 2×3 and D is size 2×2; the number of columns in B do not match the number of rows in D.

(f)

$$DB = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} (0)(1)+(-1)(0) & (0)(-1)+(-1)(2) & (0)(2)+(-1)(1) \\ (-2)(1)+(3)(0) & (-2)(-1)+(3)(2) & (-2)(2)+(3)(1) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2 & -1 \\ -2 & 8 & -1 \end{bmatrix}$$

□

The next example introduces us to polynomials involving matrices.

Example 6.4.5. Find $C^2 - 5C + 10I_2$ for $C = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$.

Solution. Just as x^2 means x times itself, C^2 denotes the matrix C times itself. We get

$$\begin{aligned}
C^2 - 5C + 10I_2 &= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -10 \\ 15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 10 \\ -15 & -10 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

□

When we started this section, we mentioned that we would temporarily consider matrices as their own entities. Now, the matrix arithmetic we have developed here will ultimately allow us to solve systems of linear equations. To that end, consider the system

$$\begin{cases} 2x + 3y + z = 4 \\ 3x + 3y + z = 2 \\ 2x + 4y + z = 5 \end{cases}$$

We may encode this system into the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 3 & 3 & 1 & 2 \\ 2 & 4 & 1 & 5 \end{array}\right]$$

Recall that the entries to the left of the vertical line come from the coefficients of the variables in the system, while those on the right consist of the associated constants. For that reason, we may form the **coefficient matrix** A, the **variable matrix** X, and the **constant matrix** B as shown below.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

We now consider the **matrix equation** $AX = B$: $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$.

Multiplying the two matrices on the left of this matrix equation results in $\begin{bmatrix} 2x + 3y + z \\ 3x + 3y + z \\ 2x + 4y + z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$.

We see that finding a solution (x, y, z) to the original system corresponds to finding a solution X for the matrix equation $AX = B$. If we think about solving the equation $ax = b$, $a \neq 0$, we would simply divide both sides by a. Is it possible to 'divide' both sides of the matrix equation $AX = B$ by the matrix A? This is the central topic of **Section 6.5**.

6.4 Exercises

1. Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

2. Can any two matrices of the same size be multiplied? If so, explain why; if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

For each pair of matrices A and B in Exercises 3 – 9, find the following, if defined.

(a) $3A$ (b) $-B$ (c) A^2

(d) $A-2B$ (e) AB (f) BA

3. $A=\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$

4. $A=\begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

5. $A=\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$

6. $A=\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

7. $A=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

8. $A=\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$, $B=\begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

9. $A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

In Exercises 10 – 23, use the following matrices to compute the indicated operation or state that the indicated operation is undefined.

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B=\begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 10 & -\frac{11}{2} & 0 \\ \frac{3}{5} & 5 & 9 \end{bmatrix} \quad D=\begin{bmatrix} 7 & -13 \\ -\frac{4}{3} & 0 \\ 6 & 8 \end{bmatrix} \quad E=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

10. $7B-4A$
11. AB
12. BA
13. $E+D$
14. ED
15. $CD+2I_2A$
16. $A-4I_2$
17. A^2-B^2
18. $(A+B)(A-B)$
19. $A^2-5A-2I_2$
20. $E^2+5E-36I_3$
21. EDC

22. $C\,D\,E$

23. $A\,B\,C\,E\,D\,I_2$

24. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Compute E_1A, E_2A, and E_3A. What effect did each of the E_i matrices have on the rows of A? Create E_4 so that its effect on A is to multiply the bottom row by -6. How would you extend this idea to matrices with more than two rows?

25. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix}$. Compare $(A+B)^2$ to $A^2 + 2AB + B^2$. Discuss with your classmates what constraints must be placed on two arbitrary matrices A and B so that both $(A+B)^2$ and $A^2 + 2AB + B^2$ exist. When will $(A+B)^2 = A^2 + 2AB + B^2$? In general, what is the correct formula for $(A+B)^2$?

In Exercises 26 – 30, consider the following definitions. A square matrix is said to be an **upper triangular matrix** if all of its entries below the main diagonal are zero and it is said to be a **lower triangular matrix** if all of its entries above the main diagonal are zero. For example,

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

is an upper triangular matrix, whereas

$$F = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

is a lower triangular matrix (zeros are allowed on the main diagonal). Discuss the following questions with your classmates.

26. Give an example of a matrix that is neither upper triangular nor lower triangular.

27. Is the product of two $n \times n$ upper triangular marices always upper triangular?

28. Is the product of two $n \times n$ lower triangular matrices always lower triangular?

29. Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, write A as LU where L is a lower triangular matrix and U is an upper triangular matrix.

30. Are there any matrices that are simultaneously upper and lower triangular?

6.5 Systems of Linear Equations: Matrix Inverses

Learning Objectives

- Find the inverse of a 2×2 or a 3×3 matrix.
- Solve a system of linear equations using an inverse matrix.

We concluded **Section 6.4** by showing how we can rewrite a system of linear equations as the matrix equation $AX = B$, where A is the coefficient matrix, B is the constant matrix, and X is the variable matrix representing a system of equations. The variable matrix X corresponds to the solution of the system. In this section, we develop the method for solving this matrix equation. To that end, consider the system

$$\begin{cases} 2x - 3y = 16 \\ 3x + 4y = 7 \end{cases}$$

To write this system as a matrix equation, we follow the procedure outlined at the end of **Section 6.4**. We find the coefficient matrix A, the variable matrix X, and the constant matrix B to be

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} 16 \\ 7 \end{bmatrix}$$

In order to motivate how we solve a matrix equation like $AX = B$, we revisit an algebraic equation, $ax = b$, where a and b are real numbers. In particular, consider the equation $3x = 5$. To solve this equation, we simply divide both sides by 3, which is the equivalent of multiplying by $\frac{1}{3}$.

$$\begin{aligned} 3x &= 5 \\ \frac{1}{3} \cdot 3x &= \frac{1}{3} \cdot 5 \\ 1 \cdot x &= \frac{1}{3} \cdot 5 \\ x &= \frac{1}{3} \cdot 5 \end{aligned}$$

Here, $\frac{1}{3}$ is the multiplicative inverse of 3 and 1 is the multiplicative identity. In general, for $ax = b$, $x = \frac{1}{a} \cdot b$ or $x = a^{-1}b$, where $a^{-1} = \frac{1}{a}$ is the multiplicative inverse for the real nonzero number a. By paying attention to the multiplicative properties of matrices, we can define an analogous process for solving a matrix equation of the form $AX = B$. We will use A^{-1}, read as 'A inverse', as the multiplicative inverse of A, if it exists. If the inverse of A exists, the product $A^{-1}A$ results in the multiplicative matrix identity, I. The identity matrix I, defined in **Section 6.4**, is a square matrix with 1's down the main diagonal and 0's elsewhere. [12]

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{multiply by } A^{-1} \text{ on same side; multiplication is not commutative!} \\ (A^{-1}A)X &= A^{-1}B && \text{Associative Property} \\ IX &= A^{-1}B \\ X &= A^{-1}B && \text{Identity Property} \end{aligned}$$

We have no guarantee that A^{-1} actually exists, but if it does we expect $AA^{-1} = A^{-1}A = I$. This implies that A and A^{-1} must be square matrices of the same size.

The Inverse of a Matrix

We begin with a formal definition of an invertible matrix.

Definition 6.9. Two $n \times n$ matrices A and B are called **inverses** if $AB = BA = I_n$. If B is the inverse of A, we denote B as A^{-1}. If A has an inverse, we refer to A as **invertible**.

Since not all matrices are square, not all matrices are invertible. However, just because a matrix is square does not guarantee it is invertible. Following is an example of two matrices that are inverses.

Example 6.5.1. Verify that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ are inverses.

Solution. We must show that $AB = I_2$ and $BA = I_2$.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} (1)(-2)+(2)\left(\frac{3}{2}\right) & (1)(1)+(2)\left(-\frac{1}{2}\right) \\ (3)(-2)+(4)\left(\frac{3}{2}\right) & (3)(1)+(4)\left(-\frac{1}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} (-2)(1)+(1)(3) & (-2)(2)+(1)(4) \\ \left(\frac{3}{2}\right)(1)+\left(-\frac{1}{2}\right)(3) & \left(\frac{3}{2}\right)(2)+\left(-\frac{1}{2}\right)(4) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Since $AB = I_2$ and $BA = I_2$, A and B are inverses.

□

[12] Here, we simply use the notation I for the identity matrix, without the subscript that indicates its size.

If a matrix A has an inverse, then that inverse is unique. To verify this, assume that both B and C are inverses of A. Then $AB = I = AC$, so $BAB = BAC$. By grouping $(BA)B = (BA)C$, we can simplify so that $IB = IC$, or $B = C$. Furthermore, if $A^{-1} = B$, by the uniqueness of the inverse matrix, then $B^{-1} = A$.

We return to the matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ from the beginning of this section and work at finding its inverse.

Knowing that the inverse will also be size 2×2, we use the variables f, g, h, and j to stand in for the missing numbers:

$$A^{-1} = \begin{bmatrix} f & g \\ h & j \end{bmatrix}$$

Now, since $AA^{-1} = I_2$, we have

$$\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} f & g \\ h & j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2f-3h & 2g-3j \\ 3f+4h & 3g+4j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives rise to two systems of linear equations.

$$\begin{cases} 2f-3h=1 \\ 3f+4h=0 \end{cases} \qquad \begin{cases} 2g-3j=0 \\ 3g+4j=1 \end{cases}$$

We encode each system into an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array}\right]$$

At this point, we could use Gaussian Elimination to solve each system. Instead, we notice that the left sides of the augmented matrices are equal. This allows us to combine the two augmented matrices into a single matrix, as shown below, and hence solve both systems at the same time.

$$\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

We use row operations to put our augmented matrix into reduced row echelon form.

$$\text{Replace } R_1 \text{ with } \frac{1}{2}R_1 \rightarrow \qquad \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\text{Replace } R_2 \text{ with } -3R_1 + R_2 \rightarrow \qquad \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{17}{2} & -\frac{3}{2} & 1 \end{array}\right]$$

$$\text{Replace } R_2 \text{ with } \frac{2}{17}R_2 \rightarrow \qquad \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{17} & \frac{2}{17} \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } \frac{3}{2}R_2 + R_1 \rightarrow \qquad \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{17} & \frac{3}{17} \\ 0 & 1 & -\frac{3}{17} & \frac{2}{17} \end{array}\right]$$

Separating out the solutions of the original systems, we have the two matrices.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array}\right]$$

From the first, we get $f = \frac{4}{17}$ and $h = -\frac{3}{17}$. The second matrix tells us that $g = \frac{3}{17}$ and $j = \frac{2}{17}$. The resulting inverse matrix is

$$A^{-1} = \begin{bmatrix} f & g \\ h & j \end{bmatrix} = \begin{bmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{bmatrix}$$

We leave it to the reader to check that $AA^{-1} = A^{-1}A = I_2$. Now, returning to our discussion at the beginning of this section, we know

$$X = A^{-1}B = \begin{bmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{bmatrix} \begin{bmatrix} 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Since $X = \begin{bmatrix} x \\ y \end{bmatrix}$, our final solution to the system is $(x, y) = (5, -2)$.

We have the following strategy for finding inverses of square matrices, noting that some square matrices do not have inverses. Square matrices without inverses are called **singular**, while matrices with inverses are referred to as **nonsingula**r.

Finding the Inverse of an $n \times n$ Matrix A

1. Form the augmented matrix $[A \mid I_n]$.
2. Use row operations to transform the matrix A to its reduced row echelon form.
3. If the identity matrix does not appear to the left of the vertical bar, the matrix does not have an inverse. Otherwise, the matrix is invertible and the inverse matrix will appear to the right of the vertical bar: $[I_n \mid A^{-1}]$.

Example 6.5.2. Given the 2×2 matrix A, find its inverse, if it exists.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution. We form the matrix $[A \mid I_2]$ and proceed with row operations to transform the matrix to reduced row echelon form.

$$\text{Matrix } [A \mid I_2] \to \quad \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

$$\text{Replace } R_2 \text{ with } -2R_1 + R_2 \to \quad \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } 2R_2 + R_1 \to \quad \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Noting that the identity matrix appears to the left of the vertical bar, we have $A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$.

□

Example 6.5.3. Given the 2×2 matrix A, find its inverse, if it exists.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Solution. Beginning with the matrix $[A \mid I_2]$, we proceed with row operations, in an attempt to rewrite the matrix in reduced row echelon form.

$$\text{Matrix } [A \mid I_2] \rightarrow \quad \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$$\text{Switch } R_1 \text{ and } R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

$$\text{Replace } R_2 \text{ with } -3R_1 + R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The matrix is in reduced row echelon form, but the identity matrix does not appear to the left of the vertical bar. Thus, this matrix does not have an inverse.

□

Before trying a 3×3 matrix, we develop a formula that can be used in finding inverses of 2×2 matrices. Assume we begin with the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then follow the procedure, as above, in determining the inverse.

$$\text{Matrix } [A \mid I_2] \rightarrow \quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\text{Replace}^{13}\ R_1 \text{ with } \frac{1}{a}R_1 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\text{Replace}^{14}\ R_2 \text{ with } -cR_1 + R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\text{Replace}^{15}\ R_2 \text{ with } \frac{a}{ad-bc} \cdot R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } -\frac{b}{a} \cdot R_2 + R_1 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

We now have a formula for A^{-1}:

---

[13] If $a \neq 0$.
[14] If $ad - bc = 0$, we will not get the identity matrix on the left side.
[15] If $ad - bc \neq 0$.

$$A^{-1}=\begin{bmatrix}\dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc}\\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc}\end{bmatrix} \text{ or, more simply, } A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} \text{ if } ad-bc\neq 0.$$

Before moving on, let's look at the meaning of $ad-bc\neq 0$ from a geometric perspective. As we saw earlier, if we can find A^{-1} then the system $\begin{cases}ax+by=m\\ cx+dy=n\end{cases}$ has the unique solution $\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}\begin{bmatrix}m\\ n\end{bmatrix}$. Solving this system is equivalent to finding the point(s) of intersection of two lines. First, we consider the case $b\neq 0$ and $d\neq 0$. The slope of the line $ax+by=m$ is $-\frac{a}{b}$ since $y=-\frac{a}{b}x+\frac{m}{b}$. Similarly, the slope of the line $cx+dy=n$ is $-\frac{c}{d}$. These two lines have a unique point of intersection if and only if their slopes are not equal; that is, $-\frac{a}{b}\neq-\frac{c}{d}$, which is equivalent to $ad-bc\neq 0$. In the case that $b=0$ or $d=0$, it can be verified that the two lines still have a unique point of intersection, if and only if $ad-bc\neq 0$. So, A^{-1} exists if and only if the two lines have a unique point of intersection, or if $ad-bc\neq 0$.

We will revisit the formula for A^{-1} in our next section as it ties in nicely with determinants. For now, we look at an example of finding the inverse of a 3×3 matrix before moving on to solving systems of equations with inverse matrices.

Example 6.5.4. Find the inverse of the given matrix, if it exists.

$$A=\begin{bmatrix}2 & 3 & 1\\ 3 & 3 & 1\\ 2 & 4 & 1\end{bmatrix}$$

Solution. We start with the matrix $[A \mid I_3]$ and apply row operations to get it into reduced row echelon form.

$$\text{Matrix } [A \mid I_3]\rightarrow \quad \left[\begin{array}{ccc|ccc}2 & 3 & 1 & 1 & 0 & 0\\ 3 & 3 & 1 & 0 & 1 & 0\\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]$$

$$\text{Replace } R_1 \text{ with } \frac{1}{2}R_1\rightarrow \quad \left[\begin{array}{ccc|ccc}1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 3 & 3 & 1 & 0 & 1 & 0\\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]$$

Replace R_2 with $-3R_1+R_2$ and R_3 with $-2R_1+R_3 \rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$

Switch R_2 and $R_3 \rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right]$

Replace R_3 with $\frac{3}{2}R_2+R_3 \rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & -3 & 1 & \frac{3}{2} \end{array}\right]$

Replace R_3 with $-2R_3 \rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$

Replace R_1 with $-\frac{3}{2}R_2+R_1 \rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 2 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$

Replace R_1 with $-\frac{1}{2}R_3+R_1 \rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$

With the matrix in reduced row echelon form, and I_3 to the left of the vertical bar, we find the inverse is

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

□

Using Matrix Inverses to Solve Systems of Linear Equations

Recall from the beginning of this section that $AX = B \Rightarrow X = A^{-1}B$, only if A^{-1} exists. If an $n \times n$ linear system (a system with n linear equations and n variables) has a unique solution, this method may be applied to solve the system.

Example 6.5.5. Solve the system of equations using an inverse matrix, if possible.

$$\begin{cases} 3x+8y=5 \\ 4x+11y=7 \end{cases}$$

Solution. We write the system in terms of the coefficient matrix A, the variable matrix X, and the constant matrix B.

$$A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Now that we have the matrices that comprise the matrix equation $AX=B$, we solve for X using the formula $X=A^{-1}B$. First, we need to calculate A^{-1}. We use the formula derived earlier to calculate the inverse of a 2×2 matrix,[16] $A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, from which $ad-bc=3(11)-8(4)=1$. Since $ad-bc\neq 0$, the inverse exists. We calculate the inverse as follows.

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

Now we are ready to solve the system using this inverse matrix.

$$\begin{aligned} X &= A^{-1}B \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution is $x=-1$ and $y=1$, or $(-1,1)$.

□

In the solution $X=A^{-1}B$, be sure to place A^{-1} to the left of B since matrix multiplication is not commutative. We move on to a last example. Part (a) may look familiar. This is the system of equations presented at the end of **Section 6.4**.

Example 6.5.6 Solve the systems of equations, if possible, using an inverse matrix.

[16] On your own, try the method from **Example 6.5.2** for finding the inverse. Since it does not require memorizing a formula, that method may be preferable.

(a) $\begin{cases} 2x+3y+z=4 \\ 3x+3y+z=2 \\ 2x+4y+z=5 \end{cases}$ (b) $\begin{cases} 2x+3y+z=11 \\ 3x+3y+z=-2 \\ 2x+4y+z=15 \end{cases}$

Solution. We note that the coefficient matrix is the same for each of these systems:

$$A=\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

In **Example 6.5.4**, we found the inverse of this matrix to be

$$A^{-1}=\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

The only difference between the system in part (a) and the system in part (b) is the constants in the matrix B for the associated matrix equation $AX=B$. We solve each system using the formula $X=A^{-1}B$.

(a) $X=A^{-1}B=\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}=\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Our solution is $(-2,1,5)$.

(b) $X=A^{-1}B=\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 11 \\ -2 \\ 15 \end{bmatrix}=\begin{bmatrix} -13 \\ 4 \\ 25 \end{bmatrix}$. We get $(-13,4,25)$.

□

6.5 Exercises

1. In a previous section, we showed examples in which matrix multiplication is not commutative, that is, $AB \neq BA$. Explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$.

2. Does every 2×2 matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

In Exercises 3 – 8, verify that the matrix A is the inverse of the matrix B.

3. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5. $A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & -\frac{4}{35} \end{bmatrix}$

6. $A = \begin{bmatrix} -2 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 1 & 6 & 9 \end{bmatrix}$, $B = \frac{1}{4}\begin{bmatrix} 6 & 0 & -2 \\ 17 & -3 & -5 \\ -12 & 2 & 4 \end{bmatrix}$

In Exercises 9 – 22, find the inverse of the matrix or state that the matrix is singular (not invertible).

9. $A = \begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix}$

10. $B = \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

11. $C = \begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix}$

12. $D = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

13. $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

14. $F = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

15. $G = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

16. $H = \begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

17. $J = \begin{bmatrix} 1 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & 2 \end{bmatrix}$

18. $K = \begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

19. $L=\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

20. $M=\begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

21. $N=\begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

22. $P=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

In Exercises 23 – 30, use a matrix inverse to solve the system of linear equations.

23. $\begin{cases} 3x+7y=26 \\ 5x+12y=39 \end{cases}$

24. $\begin{cases} 3x+7y=0 \\ 5x+12y=-1 \end{cases}$

25. $\begin{cases} 3x+7y=-7 \\ 5x+12y=5 \end{cases}$

26. $\begin{cases} 5x-6y=-61 \\ 4x+3y=-2 \end{cases}$

27. $\begin{cases} 8x+4y=-100 \\ 3x-4y=1 \end{cases}$

28. $\begin{cases} 3x-2y=6 \\ -x+5y=-2 \end{cases}$

29. $\begin{cases} -3x-4y=9 \\ 12x+4y=-6 \end{cases}$

30. $\begin{cases} -2x+3y=\dfrac{3}{10} \\ -x+5y=\dfrac{1}{2} \end{cases}$

In Exercises 31 – 33, use the inverse of M from **Exercise 20** to solve the system of linear equations.

31. $\begin{cases} 3x+4z=1 \\ 2x-y+3z=0 \\ -3x+2y-5z=0 \end{cases}$

32. $\begin{cases} 3x+4z=0 \\ 2x-y+3z=1 \\ -3x+2y-5z=0 \end{cases}$

33. $\begin{cases} 3x+4z=0 \\ 2x-y+3z=0 \\ -3x+2y-5z=1 \end{cases}$

34. Solve the following system of linear equations using an inverse matrix.

$$\begin{cases} x_1+2x_2+3x_4=2 \\ x_2+x_3+x_4=-1 \\ x_2+x_4=5 \\ x_1+2x_2+2x_4=0 \end{cases}$$

The inverse matrix is provided for you as follows: $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 0 & -2 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

35. Solve the following system of linear equations using an inverse matrix.

$$\begin{cases} x+y-3w=7 \\ x-2z=1 \\ 2y-z+w=4 \\ 2x+3y-2w=-3 \end{cases}$$

The inverse matrix is provided for you as follows: $\begin{bmatrix} 1 & 1 & 0 & -3 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 3 & 0 & -2 \end{bmatrix}^{-1} = \dfrac{1}{21}\begin{bmatrix} -14 & 7 & -14 & 14 \\ 2 & -4 & 8 & 1 \\ -7 & -7 & -7 & 7 \\ -11 & 1 & -2 & 5 \end{bmatrix}$.

36. Solve the following system of linear equations using an inverse matrix.

$$\begin{cases} x+y+z+w=5 \\ x-y+z-w=2 \\ x+2y+3z+4w=0 \\ x-2y+3z-4w=9 \end{cases}$$

The inverse matrix is provided for you as follows: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 1 & -2 & 3 & -4 \end{bmatrix}^{-1} = \dfrac{1}{4}\begin{bmatrix} 3 & 3 & -1 & -1 \\ 4 & -4 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -2 & 2 & 1 & -1 \end{bmatrix}$.

37. Matrices can be used in cryptography. Suppose we wish to encode the message ‘BIGFOOT LIVES’. We start by assigning a number to each letter of the alphabet, say $A=1$, $B=2$, and so on. We reserve 0 to act as a space. Hence, our message ‘BIGFOOT LIVES’ corresponds to the string of numbers ‘2, 9, 7, 6, 15, 15, 20, 0, 12, 9, 22, 5, 19’. To encode this message, we use an invertible matrix. Any invertible matrix will do, but for this exercise, we choose

$$A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$$

Since A is a 3×3 matrix, we encode our message string into a matrix M with 3 rows. To do this, we take the first three numbers, 2, 9, 7, and make them our first column, the next three numbers, 6, 15, 15, and make them our second column, and so on. We put 0’s to round out the matrix.

$$M=\begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$$

To encode the message, we find the product AM.

$$AM = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix}$$

So our coded message is '12, 1, −12, 42, 3, −42, 100, 36, −152, −23, 39, −46, 38, 57, −133'. To decode this message, we start with this string of numbers, construct a message matrix as we did earlier (we should get the matrix AM again) and then multiply by A^{-1}.

(a) Find A^{-1}.

(b) Use A^{-1} to decode the message and check that this method actually works.

(c) Decode the message '14, 37, −76, 128, 21, −151, 31, 65, −140'.

(d) Choose another invertible matrix and encode and decode your own message.

6.6 Systems of Linear Equations: Determinants

Learning Objectives

- Find the determinant of a 2×2 or 3×3 matrix.
- Solve a system of linear equations using Cramer's Rule.

In this section we assign to each square matrix A a real number, called the **determinant of A**, which will lead to another technique for solving consistent independent systems of linear equations. There are two commonly used notations for the determinant of a matrix A: '$\det(A)$' or '$|A|$'. The second notation, $|A|$, can be troublesome when confused with the absolute value, but is useful in providing a quick, easy, method for denoting the determinant of a matrix.

Finding the Determinant of a 2×2 Matrix

While the definition will show up a bit later, to help us get started right away finding determinants, we introduce the following formula for finding determinants of 2×2 matrices.

Formula 6.1. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$.

We may also indicate the determinant using the notation $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Example 6.6.1. Compute the determinant of the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$.

Solution. Using **Formula 6.1**, we find the determinant of A as follows.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} \\ &= (4)(1) - (-3)(2) \\ &= 4 + 6 \\ &= 10 \end{aligned}$$

□

Recall that, in **Section 6.5**, we derived a formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. From **Formula 6.1**, we can now modify that formula, using $\det(A) = ad - bc$:

$$\text{If } \det(A) \neq 0 \text{ then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, we can use the determinant of a 2×2 matrix in finding its inverse. Note that when the determinant is zero, the inverse does not exist.

Finding the Determinant of an $n \times n$ Matrix

While the technique discussed here may be applied to any $n \times n$ matrix where $n > 1$, we will focus on 3×3 matrices. We begin by introducing a notation for the entries in a 3×3 matrix A. We denote each entry as a_{ij} where *i* is the row and *j* is the column where that entry resides. For example, a_{23} is the entry in row 2 and column 3, as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note that any matrix may be written in a similar manner by increasing or decreasing the number of rows and columns. We move on to a couple of definitions.

Definition 6.10. Given an $n \times n$ matrix A where $n > 1$,

- the **minor** M_{ij} of the entry a_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix formed by deleting row i and column j.
- the **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Applying the definition, we determine the minor of an entry by deleting the row and column in which that entry appears and then finding the determinant of the resulting matrix. Consider the matrix A, below.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

Entry a_{ij}	Delete row i and column j	M_{ij}
$a_{11} = 3$	$\begin{bmatrix} \boxed{\cancel{3}} & \cancel{1} & \cancel{2} \\ \cancel{0} & -1 & 5 \\ \cancel{2} & 1 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}$	$M_{11} = \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) = (-1)(4)-(5)(1) = -9$
$a_{23} = 5$	$\begin{bmatrix} 3 & 1 & \cancel{2} \\ \cancel{0} & -\cancel{1} & \boxed{\cancel{5}} \\ 2 & 1 & \cancel{4} \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$	$M_{23} = \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right) = (3)(1)-(1)(2) = 1$

Note that there are seven additional minors (M_{12}, M_{13}, M_{21}, M_{22}, M_{31}, M_{32}, and M_{33}) that can be found in a similar manner. To determine the cofactor, C_{ij}, of entry a_{ij}, we find the minor and multiply it by 1 or −1, depending on whether the sum of i and j is even or odd, respectively. Another way to remember the sign $(-1)^{i+j}$ is with the following 'checkerboard' sign pattern.

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

We find C_{11} and C_{23} as follows.

$$\begin{aligned} C_{11} &= (-1)^{1+1} M_{11} \\ &= (1)(-9) \\ &= -9 \end{aligned} \qquad \begin{aligned} C_{23} &= (-1)^{2+3} M_{23} \\ &= (-1)(1) \\ &= -1 \end{aligned}$$

Note that, for C_{11}, $(-1)^{1+1} = (-1)^2 = +1$ matches the sign is the first row, first column of the sign pattern shown above. For C_{23}, $(-1)^{2+3} = (-1)^5 = -1$ matches the sign in the second row, third column.

We are now ready to define the determinant of an $n \times n$ matrix, $n > 1$. Because of the recursive[17] nature of calculating the determinants, we must have a starting point and thus define the determinant of a 1×1 matrix as $\det([a_{11}]) = a_{11}$. The definition for $n > 1$ follows.

Definition 6.11. Given an $n \times n$ matrix A where $n > 1$, the **determinant** of A is obtained by choosing a single row or column of A and multiplying each entry of that row or column by its respective cofactor, then adding all such products from the chosen row or column.

We refer to the process described in **Definition 6.11** as **expanding along the *i* th row**, or **down the *j* th column**. It can be shown that the value of the determinant is independent of the row or column chosen. The best way to understand the definition is through examples that we show below.

Definition 6.11 can be used to verify **Formula 6.1**. We find the determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by expanding along the first row. Noting that $a_{11} = a$ and $a_{12} = b$,

[17] We will talk more about 'recursive' in **Section 7.1**.

$$\det(A)=\det\left(\begin{bmatrix}\boldsymbol{a} & \boldsymbol{b}\\ c & d\end{bmatrix}\right)=\boldsymbol{a}\cdot C_{11}+\boldsymbol{b}\cdot C_{12}$$

Now, $C_{11}=(-1)^{1+1}M_{11}=M_{11}$ and $C_{12}=(-1)^{1+2}M_{12}=-M_{12}$ so $\det(A)=a\cdot M_{11}-b\cdot M_{12}$. We determine M_{11} and M_{12} as follows.

$a_{11}=a$	$\begin{bmatrix}\boxed{\cancel{a}} & \cancel{b}\\ \cancel{c} & d\end{bmatrix}\to[d]$	$M_{11}=\det([d]) =d$

$a_{12}=b$	$\begin{bmatrix}\cancel{a} & \boxed{\cancel{b}}\\ c & \cancel{d}\end{bmatrix}\to[c]$	$M_{12}=\det([c]) =c$

It follows that $\det\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)=ad-bc$, verifying **Formula 6.1**.

Example 6.6.2. Compute the determinant of the matrix $A=\begin{bmatrix}3 & 1 & 2\\ 0 & -1 & 5\\ 2 & 1 & 4\end{bmatrix}$.

Solution. We choose to expand along the first row. To get a general idea of what this entails, we find

$$\begin{aligned}|A|&=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\\ &=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}\end{aligned}$$

Noting that $C_{11}=(-1)^{1+1}M_{11}=M_{11}$, $C_{12}=(-1)^{1+2}M_{12}=-M_{12}$ and $C_{13}=(-1)^{1+3}M_{3}=M_{3}$,

$$|A|=a_{11}M_{11}-a_{12}M_{12}+a_{13}M_{13}$$

We expand along the first row of our matrix $A=\begin{bmatrix}3 & 1 & 2\\ 0 & -1 & 5\\ 2 & 1 & 4\end{bmatrix}$ to get

$$\begin{aligned}|A|&=(3)\begin{vmatrix}-1 & 5\\ 1 & 4\end{vmatrix}-(1)\begin{vmatrix}0 & 5\\ 2 & 4\end{vmatrix}+(2)\begin{vmatrix}0 & -1\\ 2 & 1\end{vmatrix}\\ &=3\left[(-1)(4)-(5)(1)\right]-\left[(0)(4)-(5)(2)\right]+2\left[(0)(1)-(-1)(2)\right]\\ &=3(-9)-(-10)+2(2)\\ &=-13\end{aligned}$$

Alternate Solution. Just to show that we may use any row or column, and noting that it often saves time to expand along a row or column containing the most zeros, let's find the determinant by expanding down the first column.

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{vmatrix}$$
$$= +(3)\begin{vmatrix} -1 & 5 \\ 1 & 4 \end{vmatrix} - (0)\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} + (2)\begin{vmatrix} 1 & 2 \\ -1 & 5 \end{vmatrix}$$
$$= 3[-4-5] - 0[4-2] + 2[5-(-2)]$$
$$= -13$$

□

Note that we relied on **Formula 6.1** to compute three determinants of 2×2 matrices as part of the solution process in finding the determinant of a 3×3 matrix. Evaluating the determinant of a 4×4 matrix requires computing the determinants of four 3×3 matrices, and evaluating the determinant of a 5×5 matrix requires computing determinants of five 4×4 matrices. As you can see, our method of evaluating determinants quickly gets out of hand without the aid of technology. There is an alternate method that may be used to find the determinant of a 3×3 matrix, as shown below, with the warning that this method will not work for matrices larger than 3×3.

Say we want to find the determinant of the following matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We may find the determinant as follows:

1. Augment the matrix A with the first two columns.

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right|$$

2. Starting in the upper left corner, multiply the entries down the first diagonal. Moving to the right, add the result to the product of entries down the second diagonal. Add this result to the product of the entries down the third diagonal.
3. Starting in the lower left corner, subtract the product of entries up the first diagonal. Moving to the right, subtract, from this result, the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.

$$\left|\begin{array}{ccc|cc} \bcancel{a_{11}} & \bcancel{a_{12}} & \bcancel{a_{13}} & a_{11} & a_{12} \\ a_{21} & \bcancel{a_{22}} & \bcancel{a_{23}} & \bcancel{a_{21}} & a_{22} \\ a_{31} & a_{32} & \bcancel{a_{33}} & \bcancel{a_{31}} & \bcancel{a_{32}} \end{array}\right| \qquad \left|\begin{array}{ccc|cc} a_{11} & a_{12} & \cancel{a_{13}} & \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} & \cancel{a_{23}} & \cancel{a_{21}} & a_{22} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} & a_{31} & a_{32} \end{array}\right|$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Example 6.6.3. Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ using the alternate method.

Solution. We begin by augmenting the matrix with the first two columns.

$$\left|\begin{array}{ccc|cc} 3 & 1 & 2 & 3 & 1 \\ 0 & -1 & 5 & 0 & -1 \\ 2 & 1 & 4 & 2 & 1 \end{array}\right|$$

We proceed by multiplying then adding entries down diagonals, and multiplying then subtracting entries up diagonals.

$$\begin{aligned} |A| &= (3)(-1)(4) + (1)(5)(2) + (2)(0)(1) - (2)(-1)(2) - (1)(5)(3) - (4)(0)(1) \\ &= -12 + 10 + 0 + 4 - 15 - 0 \\ &= -13 \end{aligned}$$

□

There is no analogous 'shortcut method' for computing the determinant of a 4×4 or larger matrix. We use the definition to calculate the determinant of the 4×4 matrix in the next example.

Example 6.6.4. Find the determinant of the matrix $G = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & -1 \\ 2 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$.

Solution. Noting that the fourth column has mostly zeros, we expand along the fourth column as follows:

$$\begin{aligned} |G| &= \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & -1 \\ 2 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{vmatrix} \\ &= -(0)\begin{vmatrix} 0 & 3 & 0 \\ 2 & -1 & 4 \\ 0 & 1 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 0 & 1 & -1 \end{vmatrix} - (0)\begin{vmatrix} 1 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 1 & -1 \end{vmatrix} + (0)\begin{vmatrix} 1 & 0 & 3 \\ 0 & 3 & 0 \\ 2 & -1 & 4 \end{vmatrix} \end{aligned}$$

$$|G| = 0 - \begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 0 & 1 & -1 \end{vmatrix} - 0 + 0 \qquad \text{(continuing)}$$

$$= -\left[+(1)\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} - (0)\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} + (3)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \right] \qquad \text{expanding along the first row}$$

$$= -[-3 - 0 + 6]$$

$$= -3$$

□

Cramer's Rule

We next introduce a theorem that enables us to solve a system of linear equations by means of determinants only. The theorem is stated in full generality, using numbered variables x_1, x_2, etc., instead of the more familiar letters x, y, z, etc. The proof of the general case is best left to a course in Linear Algebra.

Theorem 6.6. Cramer's Rule: Suppose $AX = B$ is the matrix form of a system of n linear equations in n variables where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for variables $x_1, x_2, \ldots, x_n$ is given by

$$x_j = \frac{\det(A_j)}{\det(A)}$$

where A_j is the matrix A whose jth column has been replaced by the constant matrix B, for $j = 1, 2, \cdots, n$.

In words, Cramer's Rule tells us we can solve for each variable, one at a time, by finding the ratio of the determinant of A_j to that of the determinant of the coefficient matrix. The matrix A_j is found by replacing the column in the coefficient matrix that holds the coefficients of x_j with the constants of the system. To understand why this works for solving 2×2 systems of equations, consider the following.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \longrightarrow \underbrace{\begin{cases} adx + bdy = ed \\ -bcx - bdy = -bf \end{cases}}_{(ad - bc)x = ed - bf}$$

Thus, $x = \dfrac{ed - bf}{ad - bc} = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and, similarly, $y = \dfrac{af - ce}{ad - bc} = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ if $ad - bc \neq 0$.

The condition $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$ is equivalent to the lines $ax + by = e$ and $cx + dy = f$ having a unique point of intersection, as discussed in **Section 6.5**. If the two lines $ax + by = e$ and $cx + dy = f$ are parallel, they do not have a unique point of intersection, the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ does not have an inverse, and the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$.

The following two examples demonstrate Cramer's Rule.

Example 6.6.5. Use Cramer’s Rule to solve for x_1 and x_2.

$$\begin{cases} 2x_1 - 3x_2 = 4 \\ 5x_1 + x_2 = -2 \end{cases}$$

Solution. Writing this system in matrix form, we find

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

To find the matrix A_1, we remove the column of the coefficient matrix A that holds the coefficients of x_1 and replace it with the corresponding entries in B. Likewise, we replace the column of A that corresponds to the coefficients of x_2 with the constants to form the matrix A_2. This yields

$$A_1 = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 5 & -2 \end{bmatrix}$$

Computing determinants, we get $\det(A) = 17$, $\det(A_1) = -2$, and $\det(A_2) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

□

Example 6.6.6. Use Cramer's Rule to solve for z.

$$\begin{cases} 2x-3y+z=-1 \\ x-y+z=1 \\ 3x-4z=0 \end{cases}$$

Solution. To get started, we determine the matrices A, X, and B.

$$A=\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \qquad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Then, in solving for z, we find the matrix[18] A_z that results from replacing the column containing the coefficients of z in the matrix A with the constants in B.

$$A_z=\begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

We have $z=\dfrac{\det(A_z)}{\det(A)}=\dfrac{-12}{-10}=\dfrac{6}{5}$.

The reader is encouraged to find these determinants on their own, and to use Cramer's Rule to solve the system for x and y.

□

[18] We have used A_z instead of A_3 as alternate notation when variables are x, y, z, etc. instead of x_1, x_2, x_3, etc.

6.6 Exercises

1. Can we always evaluate the determinant of a square matrix? Explain why or why not.

2. Examining Cramer's Rule, explain why there is no unique solution to the system when the determinant of the coefficient matrix is 0. For simplicity, use a 2×2 matrix.

In Exercises 3 – 22, compute the determinant of the given matrix. (Most of these matrices appeared in the **6.5 Exercises**.)

3. $A=\begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix}$

4. $B=\begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

5. $C=\begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix}$

6. $D=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

7. $E=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

8. $F=\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

9. $G=\begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

10. $H=\begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

11. $J=\begin{bmatrix} 1 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & 2 \end{bmatrix}$

12. $K=\begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

13. $L=\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

14. $M=\begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

15. $N=\begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

16. $P=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

17. $Q=\begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

18. $R=\begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

19. $S = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

20. $T = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

21. $V = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 2 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

22. $W = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 10 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

In Exercises 23 – 34, use Cramer's Rule to solve the system of linear equations. (Some of these systems appeared in the **6.5 Exercises**.)

23. $\begin{cases} 3x+7y=26 \\ 5x+12y=39 \end{cases}$

24. $\begin{cases} 3x+7y=0 \\ 5x+12y=-1 \end{cases}$

25. $\begin{cases} 3x+7y=-7 \\ 5x+12y=5 \end{cases}$

26. $\begin{cases} 5x-6y=-61 \\ 4x+3y=-2 \end{cases}$

27. $\begin{cases} 8x+4y=-100 \\ 3x-4y=1 \end{cases}$

28. $\begin{cases} 3x-2y=6 \\ -x+5y=-2 \end{cases}$

29. $\begin{cases} -3x-4y=9 \\ 12x+4y=-6 \end{cases}$

30. $\begin{cases} -2x+3y=\dfrac{3}{10} \\ -x+5y=\dfrac{1}{2} \end{cases}$

31. $\begin{cases} x+y+z=3 \\ 2x-y+z=0 \\ -3x+5y+7z=7 \end{cases}$

32. $\begin{cases} 3x+y-2z=10 \\ 4x-y+z=5 \\ x-3y-4z=-1 \end{cases}$

33. $\begin{cases} x+2y-4z=-1 \\ 7x+3y+5z=26 \\ -2x-6y+7z=-6 \end{cases}$

34. $\begin{cases} -5x+2y-4z=-47 \\ 4x-3y-z=-94 \\ 3x-3y+2z=94 \end{cases}$

In Exercises 35 – 40, use Cramer's Rule to solve the system of linear equations for the indicated variable.

35. Solve for x: $\begin{cases} 4x+5y-z=-7 \\ -2x-9y+2z=8 \\ 5y+7z=21 \end{cases}$

36. Solve for y: $\begin{cases} 4x-3y+4z=10 \\ 5x-2z=-2 \\ 3x+2y-5z=-9 \end{cases}$

37. Solve for z: $\begin{cases} 4x-2y+3z=6 \\ -6x+y=-2 \\ 2x+7y+8z=24 \end{cases}$

38. Solve for x: $\begin{cases} -4x-3y-8z=-7 \\ 2x-9y+5z=\frac{1}{2} \\ 5x-6y-5z=-2 \end{cases}$

39. Solve for x_4: $\begin{cases} x_1-x_3=-2 \\ 2x_2-x_4=0 \\ x_1-2x_2+x_3=0 \\ -x_3+x_4=1 \end{cases}$

40. Solve for x_4: $\begin{cases} 4x_1+x_2=4 \\ x_2-3x_3=1 \\ 10x_1+x_3+x_4=0 \\ -x_2+x_3=-3 \end{cases}$

41. Carl's Sasquatch Attack! game card collection is a mixture of common and rare cards. Each common card is worth \$0.25 while each rare card is worth \$0.75. If his entire 117 card collection is worth \$48.75, how many of each kind of card does he own?

42. Brenda's Exotic Animal Rescue houses snakes, tarantulas and scorpions. When asked how many animals of each kind she boards, Brenda answered: 'We board 49 total animals, and I am responsible for each of their 272 legs and 28 tails.' How many of each animal does the Rescue board? (Recall: Tarantulas have 8 legs and no tails; scorpions have 8 legs and one tail; snakes have no legs and one tail.)

6.7 Partial Fraction Decomposition

Learning Objectives

- Decompose a proper rational expression with denominator of non-repeated linear factors into a sum of partial fractions.
- Decompose a proper rational expression with denominator of repeated linear factors into a sum of partial fractions.
- Decompose a proper rational expression with denominator of non-repeated irreducible quadratic factors into a sum of partial fractions.
- Decompose a proper rational expression with denominator of repeated irreducible quadratic factors into a sum of partial fractions.
- Decompose a proper rational expression with any denominator into a sum of partial fractions.

This section uses systems of linear equations to decompose rational expressions into a sum of simpler rational expressions. This process will be useful in Calculus, where simpler rational expressions are often preferred. The following is an example of what we hope to achieve.

$$\frac{x^2-x-6}{x^4+x^2}=\frac{x+7}{x^2+1}-\frac{1}{x}-\frac{6}{x^2}$$

In the past, you may have been given the right side of the equation and asked to combine the three rational expressions into one rational expression. In this case, you would have used algebraic skills, starting with determining a common denominator. The focus of this section is to develop a method by which we start with the expression on the left side and 'decompose it into **partial fractions**' to obtain the expression on the right. Essentially, we want to reverse the process of combining fractions.

The denominators of rational expressions, being polynomials, can be factored into linear or irreducible quadratic factors over the real numbers. In this section, we will only consider proper rational expressions. For decomposing rational expressions with numerators of the same or greater degree than their denominators, you would start with long division, as in some of the exercises for this section. We begin with expressions having denominators of non-repeated linear factors.

Non-Repeated Linear Factors

We are looking for two rational expressions that, when added together, result in the original expression. In this first example, the denominator is already factored so we have an idea what the partial fraction will look like. Since in this section we consider only proper rational expressions, the degree of the numerator is less than the degree of the denominator. For now, we use variables A and B to represent the unknown numerators.

Example 6.7.1. Decompose the rational expression $\frac{5x-1}{(x+1)(x-2)}$ into partial fractions.

Solution. We attempt to find values A and B for which $\frac{5x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$. The first step is to multiply through by the least common denominator to eliminate the fractions.

$$\begin{aligned} \frac{5x-1}{(x+1)(x-2)} \cdot (x+1)(x-2) &= \left(\frac{A}{x+1} + \frac{B}{x-2} \right) \cdot (x+1)(x-2) \\ 5x-1 &= \frac{A}{x+1} \cdot (x+1)(x-2) + \frac{B}{x-2} \cdot (x+1)(x-2) \\ 5x-1 &= A(x-2) + B(x+1) \end{aligned}$$

In solving for A and B, we continue by expanding the right side of the equation and collecting like terms.

$$\begin{aligned} 5x-1 &= Ax - 2A + Bx + B \\ 5x-1 &= (A+B)x + (-2A+B) \end{aligned}$$

Note that the corresponding coefficients must be the same on each side of the equation.

$$5x + (-1) = (A+B)x + (-2A+B)$$

From equating coefficients, we get the system

$$\begin{cases} A + B = 5 \\ -2A + B = -1 \end{cases}$$

This system is easily solved using either the substitution or elimination method from **Section 6.1**. We find $A = 2$ and $B = 3$ to arrive at the final answer.

$$\begin{aligned} \frac{5x-1}{(x+1)(x-2)} &= \frac{A}{x+1} + \frac{B}{x-2} \\ &= \frac{2}{x+1} + \frac{3}{x-2} \end{aligned}$$

Alternate Solution. Another technique available for decomposing some rational expressions is the Heaviside Method.[19]. For the Heaviside Method, we return to the equation $5x-1 = A(x-2) + B(x+1)$

[19] Named after Oliver Heaviside.

and look for values of x that will result in an A or B term of zero. Since $A(x-2)$ is zero for $x=2$ and $B(x+1)$ is zero for $x=-1$, we input each of these values.

$$\begin{aligned} 5x-1 &= A(x-2)+B(x+1) \\ 5(2)-1 &= A(2-2)+B(2+1) \quad \text{input } x=2 \\ 9 &= A(0)+3B \\ B &= 3 \end{aligned}$$

$$\begin{aligned} 5x-1 &= A(x-2)+B(x+1) \\ 5(-1)-1 &= A(-1-2)+B(-1+1) \quad \text{input } x=-1 \\ -6 &= A(-3)+B(0) \\ A &= 2 \end{aligned}$$

Now, we replace A and B with these results to complete the partial fraction decomposition, as shown previously.

□

A word of caution in using the Heaviside Method is that this method will not catch a mistake of having initially chosen the wrong type of decomposition. However, in many instances, it is the quicker method. Before presenting the next example, we provide an overview of partial fraction decomposition when denominators consist of non-repeated linear factors.

Partial Fraction Decomposition for Non-Repeated Linear Factors

Consider the proper rational expression $\dfrac{p(x)}{q(x)}$.

- If $q(x)$ can be factored into the distinct linear factors $q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$, then $\dfrac{p(x)}{q(x)}=\dfrac{A_1}{a_1x+b_1}+\dfrac{A_2}{a_2x+b_2}+\cdots+\dfrac{A_n}{a_nx+b_n}$, where A_1, A_2, …, A_n are constants that can be determined. In practice, we usually use A, B, C, … in place of A_1, A_2, A_3, … .
- If $q(x)$ contains irreducible quadratic and/or repeated factors, then the partial fraction decomposition of $\dfrac{p(x)}{q(x)}$ will contain other terms.[20]

Example 6.7.2. Decompose the rational expression $\dfrac{x+5}{2x^2-x-1}$ into partial fractions.

Solution. We begin by factoring the denominator to find $2x^2-x-1=(2x+1)(x-1)$. Using the variables A and B to stand in for missing constants results in

$$\frac{x+5}{2x^2-x-1}=\frac{x+5}{(2x+1)(x-1)}=\frac{A}{2x+1}+\frac{B}{x-1}$$

[20] Details and examples follow throughout this section.

We multiply through by the least common denominator.

$$\frac{x+5}{(2x+1)(x-1)}\cdot(2x+1)(x-1)=\left(\frac{A}{2x+1}+\frac{B}{x-1}\right)\cdot(2x+1)(x-1)$$
$$x+5=\frac{A}{2x+1}\cdot(2x+1)(x-1)+\frac{B}{x-1}\cdot(2x+1)(x-1)$$
$$x+5=A(x-1)+B(2x+1)$$

The next step is to expand the right side and collect like terms.

$$x+5=Ax-A+2Bx+B$$
$$x+5=(A+2B)x+(-A+B)$$

Now we look for corresponding coefficients.

$$(1)x+(5)=(A+2B)x+(-A+B)$$

From equating coefficients, we get the system

$$\begin{cases} A+2B=1 \\ -A+B=5 \end{cases}$$

The system is easily solved using the elimination method from **Section 6.1**. After adding both equations we get $3B=6$, or $B=2$. Using back substitution, we find $A=-3$. The result is

$$\frac{x+5}{2x^2-x-1}=-\frac{3}{2x+1}+\frac{2}{x-1}$$

This answer is easily checked by getting a common denominator and adding the fractions.

□

The steps for decomposing rational expressions into partial fractions are summarized below.

How to Decompose a Rational Expression into Partial Fractions

1. Create an equation by setting the original expression equal to the appropriate sum of partial fractions, with constants in the numerators to be determined.
2. Multiply both sides of the equation by the least common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Equate coefficients of terms on the left side to coefficients of like terms on the right side. Solve the resulting system of equations.
5. Use the newly found constants to rewrite the original expression as a sum of partial fractions.

Repeated Linear Factors

We may run into rational expressions that contain repeated linear factors. What this means is that, instead of $\frac{5x-1}{(x+1)(x-2)}$, we might have a rational expression such as $\frac{5x-1}{(x+1)^2}$ where a factor in the denominator has a power higher than one. Before going further, we need to determine which individual denominators could contribute to obtain $(x+1)^2$ as the least common denominator. Thinking of $(x+1)^2$ as $(x+1)(x+1)$, a so-called 'repeated' linear factor, it is possible that a term with a denominator of just $x+1$ contributes to the expression as well as $(x+1)^2$. We take a closer look in the following example.

Example 6.7.3. Decompose the expression $\frac{5x-1}{(x+1)^2}$ into partial fractions.

Solution. Starting with $\frac{5x-1}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$, we multiply both sides of the equation by the least common denominator.

$$\frac{5x-1}{(x+1)^2}\cdot(x+1)^2=\frac{A}{x+1}\cdot(x+1)^2+\frac{B}{(x+1)^2}\cdot(x+1)^2$$
$$5x-1=A(x+1)+B$$
$$5x-1=Ax+(A+B)$$

Setting corresponding coefficients equal results in the system of equations:

$$\begin{cases} A=5 \\ A+B=-1 \end{cases}$$

Substituting $A=5$, from the first equation, into the second equation, we find $B=-6$. Thus,

$$\frac{5x-1}{(x+1)^2}=\frac{5}{x+1}-\frac{6}{(x+1)^2}$$

Alternate Solution. We employ the Heaviside technique, substituting $x=-1$ into the equation $5x-1=A(x+1)+B$ to help us find B.

$$5x-1=A(x+1)+B$$
$$5(-1)-1=A(-1+1)+B \quad \text{input } x=-1$$
$$-6=B$$

In our search for A, there is no additional value of x that will zero out the B term, so we choose any value for x that has not already been substituted, say $x=0$, and we also substitute $B=-6$.

$$\begin{aligned} 5x-1 &= A(x+1)+B \\ 5(0)-1 &= A(0+1)-6 \quad \text{input } x=0,\ B=-6 \\ -1 &= A-6 \\ 5 &= A \end{aligned}$$

We have $A=5$ and $B=-6$, as we found with the traditional method.

□

Partial Fraction Decomposition for Repeated Linear Factors

Consider the proper rational expression $\frac{p(x)}{q(x)}$. If $q(x)$ has a linear factor $(ax+b)$ repeated exactly n times with $n \geq 2$, that is, $(ax+b)^n$ occurs in the factorization of $q(x)$, then the sum

$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$ occurs in the partial fraction decomposition of $\frac{p(x)}{q(x)}$ where A_1, $A_2, \ldots, A_n$ are constants that can be determined. In practice, we usually use A, B, C, … in place of A_1, A_2, A_3, … .

As seen above, for the power n in $(ax+b)^n$, the partial fraction decomposition will contain a sum of n fractions with constant numerators. In the process of decomposing rational expressions into partial fractions, we sometimes run across denominators that contain more than one type of factor, as in the following example.

Example 6.7.4. Decompose the expression $\frac{3}{x^3-2x^2+x}$ into partial fractions.

Solution. Factoring the denominator gives $x^3-2x^2+x=(x)(x^2-2x+1)=x(x-1)^2$. Noting that there are both a non-repeated linear factor and a repeated linear factor, we make sure that the sum of partial fractions includes all indicated cases.

$$\frac{3}{x^3-2x^2+x}=\frac{3}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

We multiply by the least common denominator.

$$\frac{3}{x(x-1)^2}\cdot x(x-1)^2=\left(\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right)\cdot x(x-1)^2$$

$$3=A(x-1)^2+Bx(x-1)+Cx$$

Next, we expand and collect like terms.

$$3 = A(x^2 - 2x + 1) + B(x^2 - x) + Cx$$
$$3 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$
$$3 = (A+B)x^2 + (-2A - B + C)x + A$$

The resulting system of equations is

$$\begin{cases} A + B = 0 \\ -2A - B + C = 0 \\ A = 3 \end{cases}$$

Substituting $A = 3$ into $A + B = 0$ gives $B = -3$, and substituting both for A and B in $-2A - B + C = 0$ gives $C = 3$. The final answer is

$$\frac{3}{x^3 - x^2 + x} = \frac{3}{x} - \frac{3}{x-1} + \frac{3}{(x-1)^2}$$

□

Non-Repeated Irreducible Quadratic Factors

So far, our rational expressions have contained linear factors in the denominator. Now we introduce rational expressions having denominators with irreducible quadratic factors, like $x^2 + 1$. Recall that irreducible factors are expressions that cannot be further factored over the real numbers. With linear factors, we used variables A, B, C, etc. to represent constants. For irreducible quadratic factors in the denominator, we must allow for linear expressions in each numerator so will use expressions such as $Ax + B$, $Cx + D$, etc.

Example 6.7.5. Decompose the expression $\dfrac{5x^2 - 5x + 5}{(x-2)(x^2+1)}$ into partial fractions.

Solution. Having one linear factor and one irreducible quadratic factor in the denominator, we must allow for a constant, represented by A, and a linear expression, represented by $Bx + C$, respectively.

$$\frac{5x^2 - 5x + 5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1}$$

We multiply through by the least common denominator.

$$\frac{5x^2 - 5x + 5}{(x-2)(x^2+1)} \cdot (x-2)(x^2+1) = \left(\frac{A}{x-2} + \frac{Bx+C}{x^2+1}\right) \cdot (x-2)(x^2+1)$$
$$5x^2 - 5x + 5 = A(x^2+1) + (Bx + C)(x-2)$$

Now we expand and collect like terms.

$$5x^2 - 5x + 5 = Ax^2 + A + Bx^2 - 2Bx + Cx - 2C$$
$$5x^2 - 5x + 5 = (A+B)x^2 + (-2B + C)x + (A - 2C)$$

The result is the following system of equations.

$$\begin{cases} A+B=5 \\ -2B+C=-5 \\ A-2C=5 \end{cases}$$

We solve the system of equations using a method from earlier in this chapter, and find $A=3$, $B=2$, and $C=-1$. We can now rewrite the original expression as

$$\frac{5x^2-5x+5}{(x-2)(x^2+1)}=\frac{3}{x-2}+\frac{2x-1}{x^2+1}$$

□

In **Example 6.7.5**, we had a factor of (x^2+1) in the denominator. Note that this is different than the factor $(x+1)^2$. For the factor $(x+1)^2$ in the denominator, the numerators for both partial fractions with denominators $(x+1)$ and $(x+1)^2$ will be constants, since $(x+1)$ is a linear factor. The partial fraction with denominator (x^2+1) has a linear numerator since (x^2+1) is an irreducible quadratic factor. An example of a rational expression that has both linear and irreducible quadratic factors is

$$\frac{7x^3+14x^2+9x+6}{(x+1)^2(x^2+1)}=\frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{4x+1}{x^2+1}$$

Partial Fraction Decomposition for Non-Repeated Irreducible Quadratic Factors

- Consider the proper rational expression $\frac{p(x)}{q(x)}$. If $q(x)$ can be factored into the distinct irreducible quadratic factors $q(x)=(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n)$, then $\frac{p(x)}{q(x)}=\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\frac{A_2x+B_2}{a_2x^2+b_2x+c_2}+\cdots+\frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$, where A_1, B_1, …, A_n, B_n are constants that can be determined. In practice, we usually use A, B, C, … in place of A_1, B_1, A_2, … .
- If $q(x)$ contains linear and/or repeated factors, in addition to distinct irreducible quadratic factors, then the partial fraction decomposition of $\frac{p(x)}{q(x)}$ will contain other terms.

Before moving on to **Example 6.7.6**, recall that a quadratic expression ax^2+bx+c is irreducible over the real numbers if its discriminant, b^2-4ac, is negative.

Example 6.7.6. Find the partial fraction decomposition of the expression $\dfrac{3}{x^3-x^2+x}$.

Solution. The denominator factors as $x\left(x^2-x+1\right)$. The quadratic expression, x^2-x+1, does not factor easily but we check that it is irreducible by finding the discriminant. The discriminant is $(-1)^2-4(1)(1)=-3$. Since the discriminant is negative, there are no real zeros, verifying that x^2-x+1 cannot be factored over the real numbers and is indeed irreducible. We rewrite the original expression as follows:

$$\frac{3}{x^3-x^2+x}=\frac{3}{x\left(x^2-x+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2-x+1}$$

Proceeding as usual, we multiply both sides of $\dfrac{3}{x\left(x^2-x+1\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2-x+1}$ by the least common denominator, $x\left(x^2-x+1\right)$, and get

$$\begin{aligned}
3&=A\left(x^2-x+1\right)+(Bx+C)x\\
3&=Ax^2-Ax+A+Bx^2+Cx\\
3&=(A+B)x^2+(-A+C)x+A
\end{aligned}$$

This results in the following system of equations.

$$\begin{cases} A+B=0\\ -A+C=0\\ A=3\end{cases}$$

From $A=3$ and $A+B=0$, we get $B=-3$. From $-A+C=0$, it follows that $C=A=3$. Finally,

$$\frac{3}{x^3-x^2+x}=\frac{3}{x}+\frac{-3x+3}{x^2-x+1}$$

□

Repeated Irreducible Quadratic Factors

Lastly, we look at cases where an irreducible quadratic factor has a power higher than one.

Example 6.7.7. Decompose the expression $\dfrac{x^3+5x-1}{x^4+6x^2+9}$ into partial fractions.

Solution. We recognize the denominator, x^4+6x^2+9, as being quadratic in form and factor it as $\left(x^2+3\right)^2$. Since x^2+3 is not factorable, it is irreducible. As with linear factors, we can think of

$\left(x^2+3\right)^2$ as $\left(x^2+3\right)\left(x^2+3\right)$ and recognize that a term with a denominator of x^2+3 can contribute to the expression as well as $\left(x^2+3\right)^2$. As such, we consider the following partial fraction decomposition:

$$\frac{x^3+5x-1}{x^4+6x^2+9}=\frac{x^3+5x-1}{\left(x^2+3\right)^2}=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{\left(x^2+3\right)^2}$$

After multiplying each side of the equation $\frac{x^3+5x-1}{\left(x^2+3\right)^2}=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{\left(x^2+3\right)^2}$ by the least common denominator of $\left(x^2+3\right)^2$, we find

$$x^3+5x-1=(Ax+B)\left(x^2+3\right)+Cx+D$$
$$x^3+5x-1=Ax^3+3Ax+Bx^2+3B+Cx+D$$
$$x^3+5x-1=Ax^3+Bx^2+(3A+C)x+(3B+D)$$

This results in the system

$$\begin{cases} A=1 \\ B=0 \\ 3A+C=5 \\ 3B+D=-1 \end{cases}$$

We solved this system in **Example 6.3.5**, finding $A=1$, $B=0$, $C=2$, and $D=-1$. Thus,

$$\frac{x^3+5x-1}{x^4+6x^2+9}=\frac{x}{x^2+3}+\frac{2x-1}{\left(x^2+3\right)^2}$$

□

Partial Fraction Decomposition for Repeated Irreducible Quadratic Factors

Consider the proper rational expression $\frac{p(x)}{q(x)}$. If $q(x)$ has an irreducible quadratic factor $\left(ax^2+bx+c\right)$ repeated exactly n times with $n\geq 2$, that is, $\left(ax^2+bx+c\right)^n$ occurs in the factorization of $q(x)$, then the sum $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$ occurs in the partial fraction decomposition of $\frac{p(x)}{q(x)}$ where A_1, B_1, …, A_n, B_n are constants that can be determined. In practice, we usually use A, B, C, … in place of A_1, B_1, A_2, … .

Note that, for the power n in $\left(ax^2+bx+c\right)^n$, the partial fraction decomposition will contain a sum of n fractions with linear numerators.

Example 6.7.8. Find the partial fraction decomposition of $\dfrac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}$.

Solution. The factors of the denominator are x and $\left(x^2+1\right)^2$. Since x^2+1 is not factorable, it is irreducible. We consider the following partial fraction decomposition:

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

We eliminate fractions by multiplying through by the least common denominator, $x\left(x^2+1\right)^2$, and proceed to expand the right side and collect like terms.

$$\begin{aligned}
x^4+x^3+x^2-x+1&=A\left(x^2+1\right)^2+\left(Bx+C\right)\left(x\right)\left(x^2+1\right)+\left(Dx+E\right)\left(x\right)\\
x^4+x^3+x^2-x+1&=A\left(x^4+2x^2+1\right)+\left(Bx+C\right)\left(x^3+x\right)+Dx^2+Ex\\
x^4+x^3+x^2-x+1&=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex\\
x^4+x^3+x^2-x+1&=\left(A+B\right)x^4+Cx^3+\left(2A+B+D\right)x^2+\left(C+E\right)x+A
\end{aligned}$$

The resulting system of equations is

$$\begin{cases} A+B=1 \\ C=1 \\ 2A+B+D=1 \\ C+E=-1 \\ A=1 \end{cases}$$

Right off, we have $A=1$ and $C=1$. Substituting $A=1$ into $A+B=1$, we find $B=0$. From $C=1$ and $C+E=-1$, we get $E=-2$. The substitutions $A=1$ and $B=0$ into $2A+B+D=1$ give $D=-1$. The resulting decomposition of the original equation is

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}=\frac{1}{x}+\frac{1}{x^2+1}-\frac{x+2}{\left(x^2+1\right)^2}$$

□

6.7 Exercises

1. Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example.

2. How can you check that you decomposed a partial fraction correctly?

In Exercises 3 – 16, find the partial fraction decomposition for the rational expressions with denominators that contain non-repeated linear factors.

3. $\dfrac{5x+16}{x^2+10x+24}$

4. $\dfrac{3x-79}{x^2-5x-24}$

5. $\dfrac{-x-24}{x^2-2x-24}$

6. $\dfrac{10x+47}{x^2+7x+10}$

7. $\dfrac{x}{6x^2+25x+25}$

8. $\dfrac{32x-11}{20x^2-13x+2}$

9. $\dfrac{x+1}{x^2+7x+10}$

10. $\dfrac{5x}{x^2-9}$

11. $\dfrac{10x}{x^2-25}$

12. $\dfrac{6x}{x^2-4}$

13. $\dfrac{2x-3}{x^2-6x+5}$

14. $\dfrac{4x-1}{x^2-x-6}$

15. $\dfrac{4x+3}{x^2+8x+15}$

16. $\dfrac{3x-1}{x^2-5x+6}$

In Exercises 17 – 27, find the partial fraction decomposition for the rational expressions with denominators that contain repeated linear factors.

17. $\dfrac{-5x-19}{(x+4)^2}$

18. $\dfrac{x}{(x-2)^2}$

19. $\dfrac{7x+14}{(x+3)^2}$

20. $\dfrac{-24x-27}{(4x+5)^2}$

21. $\dfrac{-24x-27}{(6x-7)^2}$

22. $\dfrac{5-x}{(x-7)^2}$

23. $\dfrac{5x+14}{2x^2+12x+18}$

24. $\dfrac{5x^2+20x+8}{2x(x+1)^2}$

25. $\dfrac{4x^2+55x+25}{5x(3x+5)^2}$

26. $\dfrac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}$

27. $\dfrac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}$

In Exercises 28 – 40, find the partial fraction decomposition for the rational expressions with denominators that contain non-repeated irreducible quadratic factors.

28. $\dfrac{4x^2+6x+11}{(x+2)(x^2+x+3)}$

29. $\dfrac{4x^2+9x+23}{(x-1)(x^2+6x+11)}$

30. $\dfrac{-2x^2+10x+4}{(x-1)(x^2+3x+8)}$

31. $\dfrac{x^2+3x+1}{(x+1)(x^2+5x-2)}$

32. $\dfrac{4x^2+17x-1}{(x+3)(x^2+6x+1)}$

33. $\dfrac{4x^2}{(x+5)(x^2+7x-5)}$

34. $\dfrac{4x^2+5x+3}{x^3-1}$

35. $\dfrac{-5x^2+18x-4}{x^3+8}$

36. $\dfrac{3x^2-7x+33}{x^3+27}$

37. $\dfrac{x^2+2x+40}{x^3-125}$

38. $\dfrac{4x^2+4x+12}{8x^3-27}$

39. $\dfrac{-50x^2+5x-3}{125x^3-1}$

40. $\dfrac{-2x^3-30x^2+36x+216}{x^4+216x}$

In Exercises 41 – 51, find the partial fraction decomposition for the rational expressions with denominators that contain repeated irreducible quadratic factors.

41. $\dfrac{2x-9}{(x^2-x)^2}$

42. $\dfrac{3x^3+2x^2+14x+15}{(x^2+4)^2}$

43. $\dfrac{x^3+6x^2+5x+9}{(x^2+1)^2}$

44. $\dfrac{x^3-x^2+x-1}{(x^2-3)^2}$

45. $\dfrac{x^2+5x+5}{(x^2+2)^2}$

46. $\dfrac{x^3+2x^2+4x}{(x^2+2x+9)^2}$

47. $\dfrac{x^2+25}{(x^2+3x+25)^2}$

48. $\dfrac{2x^3+11x^2+7x+70}{(2x^2+x+14)^2}$

49. $\dfrac{5x+2}{x(x^2+4)^2}$

50. $\dfrac{x^4+x^3+8x^2+6x+36}{x(x^2+6)^2}$

51. $\dfrac{5x^3-2x+1}{(x^2+2)^2}$

In Exercises 52 – 55, use division to decompose the improper rational expression into the sum of a polynomial and a proper rational expression.

52. $\dfrac{x^2+x-1}{x-1}$

53. $\dfrac{6x^2+5x-14}{3x+2}$

54. $\dfrac{x^3+8}{x^2+4}$

55. $\dfrac{x^3-x^2-7x+10}{x^2+2x+7}$

CHAPTER 7
SEQUENCES AND SERIES

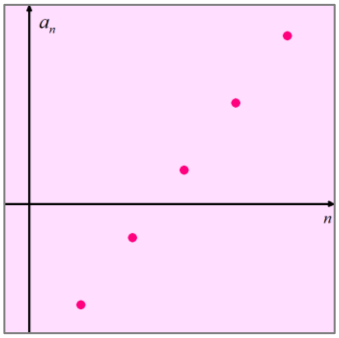


Figure 7.0. 1

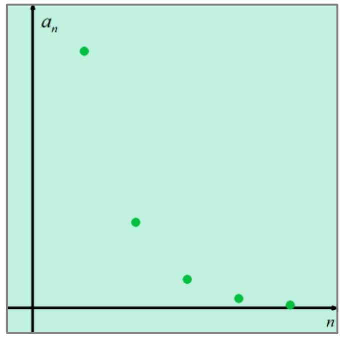


Figure 7.0. 2

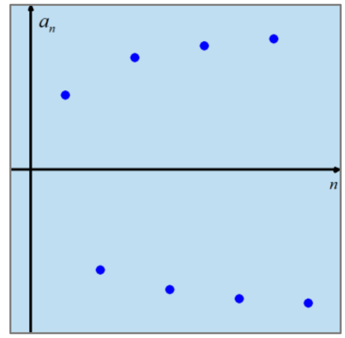


Figure 7.0. 3

Chapter Outline

Introduction

Chapter 7 is brief, but packed with new ideas that will be extended in future mathematics courses. The first two sections deal with sequences (ordered collections of terms) and series (finite or infinite sums of sequences). The majority of the two sections focuses on two types of sequences: arithmetic and geometric. By the end of these two sections, you should be able to distinguish between arithmetic and geometric patterns, write formulas for a variety of sequences, find specific terms in a sequence, and find specified sums of arithmetic and geometric sequences. Having an understanding of notation and sequence behavior, along with facility in manipulating series, will be useful in future courses. The last section focuses on binomial expansion and the binomial theorem.

Section 7.1 introduces the definition of sequences and provides a variety of examples of different types. Special attention should be paid to how ‘the next term’ is generated, as this is how some sequences are classified. In this section, special attention is given to sequences that are generated by adding a fixed amount to the previous term (arithmetic sequences) and those where the next term is generated by multiplying the previous term by a specific factor (geometric sequences). You will also learn sequence notation, how to find specific terms in sequences, and how to write formulas for arithmetic and geometric sequences.

In Section 7.2 you will learn how to find finite and infinite sums of arithmetic and geometric sequences. Again, there will be new notation introduced in this section, and information on how to use it. You will see this notation in future mathematics courses. Further, you will explore applications of summations, many of which are significantly related to our day-to-day lives (interest accumulation for savings or as part of a debt, for example).

The last section of the chapter, Section 7.3, is related to binomial expansion. Here you will see the relationship between the binomial theorem, Pascal's triangle, and the expansion of binomials. While expansion of binomials raised to the power of two or three can be readily done with simple multiplication, this section will show you a faster method and/or a way to find a specific term in the expansion for binomials raised to higher powers. Ideas in this section relate to probability and other future mathematics courses.

7.1 Sequences

Learning Objectives

- Identify number patterns.
- Recognize and use recursive and explicit formulas for sequences.
- Graph sequences.
- Identify arithmetic and geometric sequences.
- Find formulas for arithmetic and geometric sequences.

As we perceive time, life is a sequence of events. A typical macro view of a life may be: birth, schooling, job, marriage, kids, retirement, and death. Informally, any ordered collection of events or numbers is called a **sequence**. In this sense, the title of the popular books and movies 'A Series of Unfortunate Events' is mathematically incorrect; it should have been called 'A Sequence of Unfortunate Events'. This chapter is all about the mathematical definitions and applications of sequences and series.

Number Patterns

As noted above (informally), a sequence is an ordered collection. There are many sequences or number patterns in the world, some of which we can describe numerically or with a formula. The triangular numbers and the square numbers, shown below, are patterns with geometric names. Can you see why?

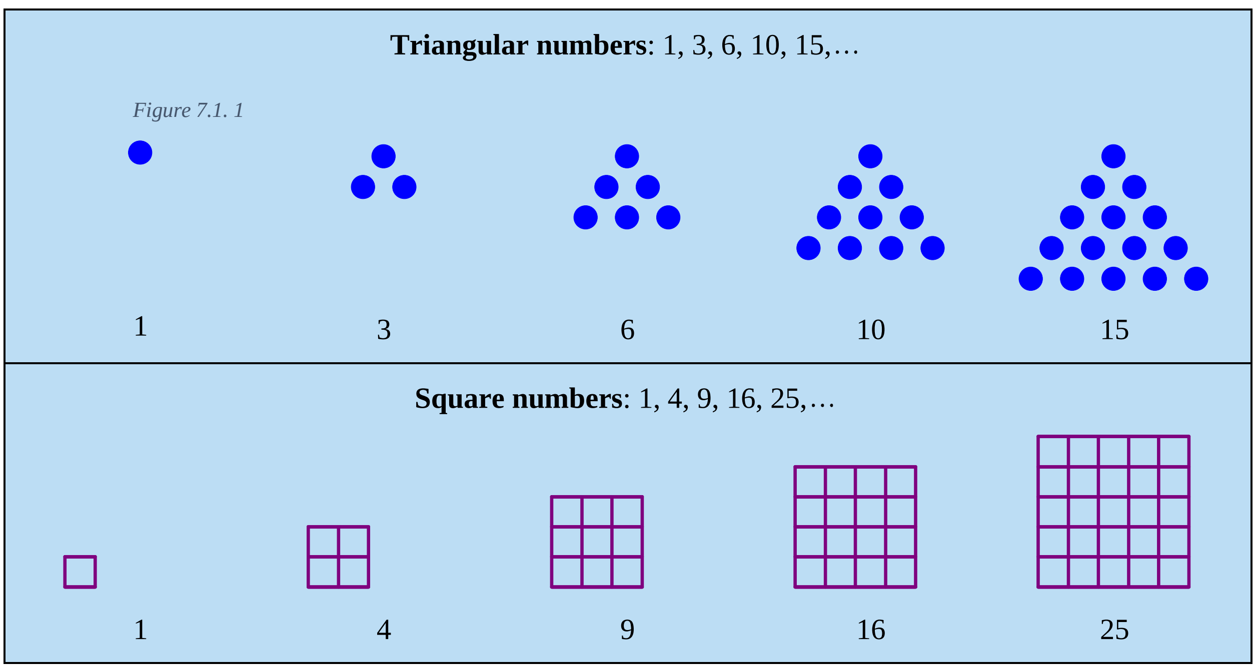


Figure 7.1. 1

Each number in a sequence is called a **term**. In the sequence $-3, -1, 1, \ldots$ the first term is -3, the second term is -1, and the third term is 1. The three dots at the end indicate this is an **infinite sequence**, meaning it continues indefinitely. It is useful to recognize the pattern of sequences and be able to describe them. The first step in writing a formula for a sequence is to understand how the next number in the sequence is generated.

Example 7.1.1. For each sequence, describe how subsequent terms may be determined. Then, state the following two terms.

a) $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ b) $1, 3, 6, 10, 15, \ldots$ c) $1, 4, 9, 16, 25, \ldots$

d) $-7, -4, -1, 2, 5, 8, \ldots$ e) $64, 32, 16, 8, 4, 2, \ldots$ f) $37, 32, 27, 22, 17, 12, \ldots$

g) $81, 54, 36, 24, 16, \ldots$ h) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ i) $-5, \frac{15}{2}, -\frac{45}{4}, \ldots$

Solution.

a) The numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called the **Fibonacci Numbers**.

$$\begin{array}{ccccccccc}
1 & 1 & \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathbf{8} & \mathbf{13} & \mathbf{21} & \ldots \\
\searrow & \downarrow\searrow & \downarrow\searrow & \downarrow\searrow & \downarrow\searrow & \downarrow\searrow & \downarrow\searrow & \downarrow & \ldots \\
 & 1+1=\mathbf{2} & 1+\mathbf{2}=\mathbf{3} & \mathbf{2}+\mathbf{3}=\mathbf{5} & \mathbf{3}+\mathbf{5}=\mathbf{8} & \mathbf{5}+\mathbf{8}=\mathbf{13} & \mathbf{8}+\mathbf{13}=\mathbf{21} & \mathbf{13}+\mathbf{21}=\mathbf{34} & \ldots
\end{array}$$

Noting that $1+1=2$, $1+2=3$, $2+3=5$, and so forth, we deduce that adding the two previous terms gives us the next term. So, to find the term after 34, we add 21 and 34 to get 55. Then, to find the term after 55, we add the previous two terms and obtain $34+55=89$.

Note: The Fibonacci sequence might have been known as early as 200 BCE in India and it appears in nature, like the number of petals of natural flowers. It is named after the Italian mathematician Leonardo of Pisa, also known as Fibonacci, and was the solution to a question he posed in his book *Liber Abaci*, published in 1202 CE. However, the most important contribution of Leonardo of Pisa to Western European mathematics was the concept of zero which he had learned about from studying the Hindu-Arabic arithmetic system while growing up in North Africa. The Fibonacci sequence has many interesting properties and we encourage the reader to investigate them.

b) The numbers $1, 3, 6, 10, 15, \ldots$ are the triangular numbers. Notice that 2 is added to the first term to get the second term; 3 is added to the second term to get the third term; 4 is added to the third term to get the fourth term. We next add 5, then 6, then 7, etc. The two terms following 15 are $15+6=21$ and $21+7=28$.

c) The numbers 1, 4, 9, 16, 25,… are the square numbers. The terms in the sequence are the squares of the consecutive natural numbers. The next two terms are $6^2 = 36$ and $7^2 = 49$.

Another way to describe this pattern is to say you start with 1, then add 3, then add 5, then add 7; that is, add consecutive odd integers.

d) For −7, −4, −1, 2, 5, 8,… we are adding the same amount, 3, in going from each term to the next. The two terms following 8 will be 11 and 14.

e) Each term in 64, 32, 16, 8, 4, 2,… is half of the previous term. Rather than dividing by two, we can multiply a term by $\frac{1}{2}$ to get the next term. The two terms following 2 are 1 and $\frac{1}{2}$.

f) In the sequence 37, 32, 27, 22, 17, 12,… we are subtracting the same amount each time. This is similar to part d; here we add −5 instead of 3. The next two terms are 7 and 2.

g) For 81, 54, 36, 24, 16,… we note that we are not adding the same amount each time as we go from term to term. We look for an amount, x, that we multiply by to take us from term to term, starting with the first two terms.

$$81x = 54$$
$$x = \frac{54}{81} = \frac{2}{3}$$

Now we must check each subsequent pair of terms, so we move on to 54 and 36.

$$54x = 36$$
$$x = \frac{36}{54} = \frac{2}{3}$$

We leave it to the reader to confirm for the rest of the terms in this sequence that we get the next term by multiplying the previous term by $\frac{2}{3}$. Following 16, we have $16 \cdot \frac{2}{3} = \frac{32}{3}$ and $\frac{32}{3} \cdot \frac{2}{3} = \frac{64}{9}$.

h) The sequence $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ is a bit different than what we have seen so far. The numerators and denominators are increasing by one each time, and the sign is alternating. A sequence in which the sign of its terms is alternating between positive and negative is referred to as an **alternating sequence**. For this sequence, the next two terms are $\frac{5}{6}$ and $-\frac{6}{7}$.

i) The sequence $-5, \frac{15}{2}, -\frac{45}{4}, \ldots$ is similar to the sequences in parts e and g in that each term is the previous term multiplied by the same constant, $-\frac{3}{2}$ in this case, and it is also like the sequence in part h in that it is an alternating sequence. The next two terms in the sequence are $\frac{135}{8}$ and $-\frac{405}{16}$.

□

In the previous example, we have written the next two terms based only on the way we see patterns. The actual pattern may be different. For example, −3, −1, 1,… may continue as −3, −1, 1, 3, 5, 7,… or as −3, −1, 1, −30, −10, 10,… , or as any of an infinite number of possibilities.

In general, there are two broad ways sequence formulas are written, recursively and explicitly. A **recursive formula** is a formula that defines each new term using one or more of its previous terms, whereas an **explicit formula** defines any term in the sequence using its position in the sequence.

We can write a sequence using function notation, as will be discussed later. However, we often use a letter, say a, along with various subscripts, to denote the terms. The subscripts are usually limited to the natural numbers, but are sometimes extended to the whole numbers or even the integers. In the sequence −3, −1, 1,… , we can write the first term as $a_1 = -3$, the second as $a_2 = -1$, the third as $a_3 = 1$, and so on. That is, $\boldsymbol{a_n}$ is the ***n*th term** of the sequence[1] for $n = 1, 2, 3, \ldots$. Using the whole numbers, the 'starting' term would be a_0 (read 'a-naught').

Recursive Formulas

Suppose we want to write a formula for −3, −1, 1, 3, 5, 7, 9,… , which is the sequence of odd integers starting with −3. While we can do so either recursively or explicitly, we begin with a recursive formula. For $n = 1, 2, 3, \ldots$, let a_n represent the *n*th term; then a_{n-1} represents the previous term and a_{n+1} represents the subsequent term. It is important to note here that the words 'previous' and 'subsequent' are relative to the *n*th term. Since, in this sequence, each term is two more than the term before it, we have $a_n = a_{n-1} + 2$. To fully describe the sequence, we also need the starting term, $a_1 = -3$. The following recursive formula defines each term in this sequence by the previous term, along with the starting term.

$$a_n = a_{n-1} + 2,\ a_1 = -3,\ n = 2, 3, 4, \ldots$$

[1] Of course, the choice of the letter *a* and the index *n* is arbitrary. We could just as well use, for example, b_n or c_k.

This may be read as 'the current term is the previous term plus 2, where we start with −3'. To find a particular term using this formula, we must also find all previous terms, as shown below in finding a_5.

$$\begin{aligned}
&a_1 = -3 \\
&a_2 = a_1 + 2 = -3 + 2 = -1 \quad \text{note: } n = 2 \Rightarrow n-1 = 1 \\
&a_3 = a_2 + 2 = -1 + 2 = 1 \quad \text{note: } n = 3 \Rightarrow n-1 = 2 \\
&a_4 = a_3 + 2 = 1 + 2 = 3 \quad \text{note: } n = 4 \Rightarrow n-1 = 3 \\
&a_5 = a_4 + 2 = 3 + 2 = 5 \quad \text{note: } n = 5 \Rightarrow n-1 = 4
\end{aligned}$$

When a formula is written recursively, we must know the previous term or terms to determine a term. For example, to find the 101st term in a sequence using its recursive formula, we would also have to find its first 100 terms. For another example, we develop a recursive formula for the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34,…. Since each term of this sequence, after the first two terms, is the sum of the previous two terms, and the first two terms are ones, we can write the following recursive formula:

$$a_n = a_{n-2} + a_{n-1},\ a_1 = 1,\ a_2 = 1,\ n = 3,\ 4,\ 5,\ldots$$

Explicit Formulas

Explicit formulas do not rely on knowing a previous term or terms. Rather, they are formulas that allow us to find any term in the sequence by simply inputting its term number. An explicit formula for the sequence of odd integers, starting with −3, is $f(n) = 2n - 5$, $n = 1,\ 2,\ 3,\ldots$. To find any term in the sequence, we evaluate using the formula.

$$\begin{aligned}
&f(1) = 2(1) - 5 = -3 \quad \text{the first term} \\
&f(2) = 2(2) - 5 = -1 \quad \text{the second term} \\
&f(3) = 2(3) - 5 = 1 \quad \text{the third term} \\
&f(4) = 2(4) - 5 = 3 \quad \text{the fourth term} \\
&f(5) = 2(5) - 5 = 5 \quad \text{the fifth term}
\end{aligned}$$

If we want the 102nd term in this sequence, we simply find $f(102) = 2(102) - 5 = 199$. Note that, although we used the traditional function notation $f(n) = 2n - 5$, we could have just as easily written the formula as $a_n = 2n - 5$. Also note that we could have started with any value of n. For example, we could define the sequence as $a_n = 2n - 7$, $n = 2,\ 3,\ 4,\ldots$.

Although there is a simple explicit formula for many sequences, sometimes the formula for a sequence is difficult to identify. For example, the explicit formula for the Fibonacci sequence is

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],\ n = 1,\ 2,\ 3,\ldots$$

We leave it to the reader to verify that this equation does indeed produce the Fibonacci sequence. In either way of expressing a sequence, we refer to a_n as the **general term** or the ***n*th term formula**.

Sequence Definition and Notation

Definition 7.1. A **sequence** is a function $f(n)$ with the domain of all natural numbers or a subset of consecutive natural numbers.[2] The subscript notation a_n, where $a_n = f(n)$, is often used for function values and is referred to as the general term or the nth term formula. It is customary to use the notation a_n, $n \geq 1$, to denote the sequence itself.

We may also write a sequence as $\{a_n\}_{n=\text{starting value}}^{\text{end value or }\infty}$ and if there is no confusion about the domain (for example, the entire set of natural numbers) we may simply write $\{a_n\}$. For example, we can write a formula for the sequence of odd natural numbers, starting with −3, in the following ways:

$$f(n) = 2n-5,\ n = 1,\ 2,\ 3,\dots$$
$$a_n = 2n-5,\ n = 1,\ 2,\ 3,\dots$$
$$\{2n-5\}_{n=1}^{\infty}$$
$$\{2n-5\}$$

Example 7.1.2. Find the first four terms of each sequence.

1. $f(n) = -3n+11,\ n = 1,\ 2,\ 3,\dots$

2. $a_n = a_{n-1}\left(\frac{1}{2}\right)^{n-2},\ a_1 = 8,\ n = 2,\ 3,\ 4,\dots$

Solution.

1. For $f(n) = -3n+11$, we find

$$f(1) = -3(1)+11 = 8$$
$$f(2) = -3(2)+11 = 5$$
$$f(3) = -3(3)+11 = 2$$
$$f(4) = -3(4)+11 = -1$$

The sequence is $8,\ 5,\ 2, -1,\dots$

2. The sequence is defined recursively with $a_1 = 8$. To find the second term, we substitute $n = 2$ and $a_1 = 8$ in the formula $a_n = a_{n-1}\left(\frac{1}{2}\right)^{n-2}$.

[2] Natural numbers (positive integers) are 1, 2, 3, $\dots$. The domain may also be a subset of any consecutive integers; for example, the whole numbers: 0, 1, 2, $\dots$. The starting n-value could be 0 or any integer.

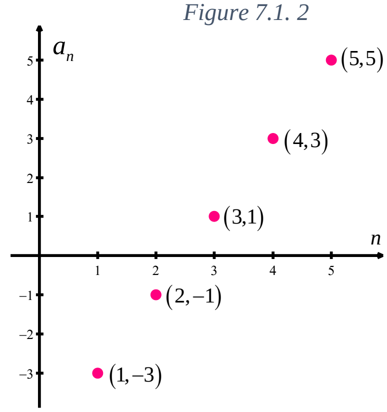
$$a_1 = 8$$
$$a_2 = a_1\left(\frac{1}{2}\right)^{2-2} = 8(1) = 8 \quad \text{substitute } n = 2 \text{ and } a_1 = 8 \text{ in the formula}$$
$$a_3 = a_2\left(\frac{1}{2}\right)^{3-2} = 8\left(\frac{1}{2}\right) = 4 \quad \text{substitute } n = 3 \text{ and } a_2 = 8 \text{ in the formula}$$
$$a_4 = a_3\left(\frac{1}{2}\right)^{4-2} = 4\left(\frac{1}{4}\right) = 1 \quad \text{substitute } n = 4 \text{ and } a_3 = 4 \text{ in the formula}$$

The sequence is $8,\ 8,\ 4,\ 1,\ldots$.

□

Because a sequence is a function, we can graph a sequence. However, the graph will only consist of isolated points as in the following graph of $a_n = 2n - 5$, for $n = 1$, 2, 3, 4, 5.

Figure 7.1. 2



Note that we can only graph a finite number of terms, although the domain in this example is all natural numbers. As a word of caution, we should not give in to the desire to connect the points together by lines or curves. It is worth observing however that, were we able to connect the points, we would have a line with slope 2. Since we do not have a line, the most we can say is that we have a constant rate of change of 2, which is consistent with the nature of a linear function, and indicative of an **arithmetic sequence**.

Example 7.1.3. Find the first four terms and graph each sequence.

1. $\left\{(-1)^{n+1}\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

2. $\left\{\dfrac{5}{3^{n-1}}\right\}_{n=1}^{\infty}$

Solution.

1. The first four terms of the explicit formula $\left\{(-1)^{n+1}\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ are found as follows.

n	1	2	3	4
$(-1)^{n+1}\dfrac{n}{n+1}$	$(-1)^{1+1}\dfrac{1}{1+1}=\dfrac{1}{2}$	$(-1)^{2+1}\dfrac{2}{2+1}=-\dfrac{2}{3}$	$(-1)^{3+1}\dfrac{3}{3+1}=\dfrac{3}{4}$	$(-1)^{4+1}\dfrac{4}{4+1}=-\dfrac{4}{5}$

This sequence, with its first four terms listed, is $\dfrac{1}{2},-\dfrac{2}{3},\dfrac{3}{4},-\dfrac{4}{5},\ldots$

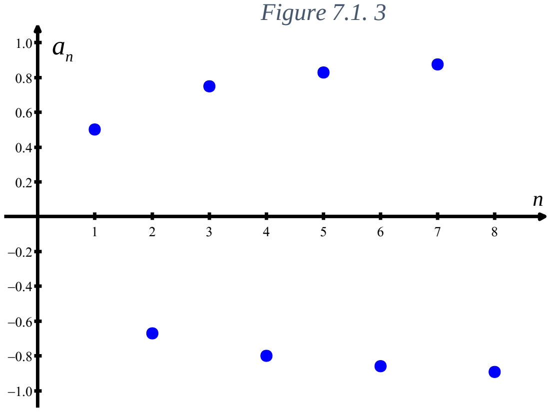
2. For $\left\{\dfrac{5}{3^{n-1}}\right\}_{n=1}^{\infty}$, we find the first four terms.

n	1	2	3	4
$\dfrac{5}{3^{n-1}}$	$\dfrac{5}{3^{1-1}}=\dfrac{5}{1}=5$	$\dfrac{5}{3^{2-1}}=\dfrac{5}{3}$	$\dfrac{5}{3^{3-1}}=\dfrac{5}{9}$	$\dfrac{5}{3^{4-1}}=\dfrac{5}{27}$

We have the sequence $5,\dfrac{5}{3},\dfrac{5}{9},\dfrac{5}{27},\ldots$

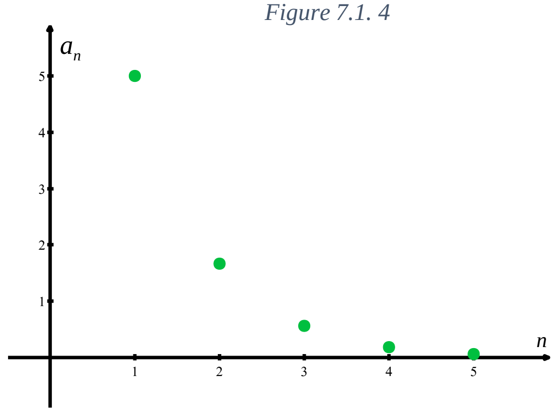
The graphs of several terms of these sequences are shown below.

Figure 7.1. 3



$$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{\infty}=\left\{\frac{1}{2},-\frac{2}{3},\frac{3}{4},-\frac{4}{5},\ldots\right\}$$

Figure 7.1. 4



$$\left\{\frac{5}{3^{n-1}}\right\}_{n=1}^{\infty}=\left\{5,\frac{5}{3},\frac{5}{9},\frac{5}{27},\ldots\right\}$$

□

As seen in the blue graph on the left, the sequence values alternate between positive and negative. Recall that any sequence with alternating positive and negative values is called an alternating sequence. The green graph, to the right, has the general shape of an exponential function. This is consistent with the nature of a geometric sequence, which we will define later.

Arithmetic and Geometric Sequences

As you may have noticed, sequences can vary greatly. We will focus in this chapter on two types of seqences: arithmetic and geometric. **Arithmetic sequences** are sequences in which each pair of consecutive terms differ by a fixed amount. Thus, the example of consecutive odd integers starting with −3 is an arithmetic sequence, having a fixed difference of 2 between terms. **Geometric sequences** are sequences in which each term is a constant multiple of the term before it. For example, the sequence in **Example 7.1.3** with terms $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \ldots$ is geometric since we multiply each term by $\frac{1}{3}$ to arrive at the next term.

Definition 7.2. The sequence $\{a_n\}$ is called

- an **arithmetic sequence** if $a_n = a_{n-1} + d$ for some constant d and all values of n and $n-1$ for which the sequence is defined. The constant d is called the **common difference**.
- a **geometric sequence** if $a_n = ra_{n-1}$ for some nonzero constant r and all values of n and $n-1$ for which the sequence is defined. The constant r is called the **common ratio**.

Note that $a_n = a_{n-1} + d$ is equivalent to $a_n - a_{n-1} = d$ and $a_n = ra_{n-1}$ is equivalent to $\frac{a_n}{a_{n-1}} = r$.

Example 7.1.4. Identify the sequences from **Example 7.1.1** as arithmetic, geometric, or neither and explain your reasoning.

Solution.

Sequence	Type	Reasoning
a) $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$	Neither	No common difference and no common ratio.
b) $1, 3, 6, 10, 15, \ldots$	Neither	No common difference and no common ratio.
c) $1, 4, 9, 16, 25, \ldots$	Neither	No common difference and no common ratio.
d) $-7, -4, -1, 2, 5, 8, \ldots$	Arithmetic	Each term is 3 more that the previous term.
e) $64, 32, 16, 8, 4, 2, \ldots$	Geometric	Each term is $\frac{1}{2}$ of the previous term.
f) $37, 32, 27, 22, 17, 12, \ldots$	Arithmetic	Each term is the previous term plus -5.
g) $81, 54, 36, 24, 16, \ldots$	Geometric	Each term is $\frac{2}{3}$ of the previous term.
h) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$	Neither	No common difference and no common ratio.
i) $-5, \frac{15}{2}, -\frac{45}{4}, \ldots$	Geometric	Each term is $-\frac{3}{2}$ of the previous term.

□

Example 7.1.5. Determine if the following sequences are arithmetic, geometric, or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

1. $a_n = 2n+1,\ n=1, 2, 3,\dots$
2. $a_n = (-1)^{n+1}\dfrac{n}{n+1},\ n=1, 2, 3,\dots$
3. $a_n = \dfrac{5}{3^{n-1}},\ n=1, 2, 3,\dots$

Solution. A good rule of thumb to keep in mind when working with sequences is 'When in doubt, write it out!' Writing out the first several terms can help you identify the pattern of the sequence.

1. The sequence $a_n = 2n+1,\ n=1, 2, 3,\dots$, generates the odd numbers $3, 5, 7, 9,\dots$. Computing the first few differences, we have $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests the sequence is arithmetic. To verify this, we find
$$\begin{aligned} a_n - a_{n-1} &= [2n+1] - [2(n-1)+1] \\ &= 2n+1-2n+2-1 \\ &= 2 \end{aligned}$$
This establishes that the sequence is arithmetic with common difference $d=2$.

2. We write out the first several terms: $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5},\dots$. We find $a_2 - a_1 = -\frac{2}{3} - \frac{1}{2} = -\frac{7}{6}$ and $a_3 - a_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$. There is not a common difference; hence, the sequence is not arithmetic. To determine if it is geometric, we compute $\frac{a_2}{a_1} = -\frac{4}{3}$ and $\frac{a_3}{a_2} = -\frac{9}{8}$. Since there is no common ratio, the sequence is not geometric either. Thus, the sequence is neither.

3. The sequence has terms $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27},\dots$ and we see right off that $\frac{5}{3} - 5 \neq \frac{5}{9} - \frac{5}{3}$ so the sequence is not arithmetic. Computing the first few ratios gives us $\frac{a_2}{a_1} = \frac{1}{3}$, $\frac{a_3}{a_2} = \frac{1}{3}$, and $\frac{a_4}{a_3} = \frac{1}{3}$. This suggests that the sequence is geometric. To verify that this is the case, we show $\frac{a_n}{a_{n-1}} = \frac{1}{3}$ for all n.
$$\frac{a_n}{a_{n-1}} = \frac{\frac{5}{3^{n-1}}}{\frac{5}{3^{(n-1)-1}}} = \frac{5}{3^{n-1}} \cdot \frac{3^{n-2}}{5} = \frac{3^{n-2}}{3^{n-1}} = \frac{3^n 3^{-2}}{3^n 3^{-1}} = \frac{3^1}{3^2} = \frac{1}{3}$$
Thus, this sequence is geometric with common ratio $r = \frac{1}{3}$.

□

As stated earlier, the focus of this chapter is arithmetic and geometric sequences. We proceed by looking more closely at the arithmetic sequence $-7, -3, 1, 5, 9, 13, 17,\dots$ to see how we might write a formula.

Noting that the common difference is 4, we can easily come up with the recursive formula $a_n = a_{n-1} + 4$, but we go a step further and look for an explicit formula.

$$\begin{array}{ccccccccccccc} \mathbf{a_1 = -7} & & \mathbf{a_2 = -3} & & \mathbf{a_3 = 1} & & \mathbf{a_4 = 5} & & \mathbf{a_5 = 9} & & \mathbf{a_6 = 13} & & \mathbf{a_7 = 17} \\ & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & \\ & \mathbf{+4} & & \mathbf{+4} & & \mathbf{+4} & & \mathbf{+4} & & \mathbf{+4} & & \mathbf{+4} & \end{array}$$

To get to the ***second*** term, we add ***one*** 4 to the first term; to get the the ***third*** term, we add ***two*** 4's to the first term; to get to the ***fourth*** term, we add ***three*** 4's to the first term; and so on.

$$\begin{aligned} a_1 &= -7 \\ a_2 &= -7+4 = -7+\mathbf{1}(4) \\ a_3 &= -7+4+4 = -7+\mathbf{2}(4) \\ a_4 &= -7+4+4+4 = -7+\mathbf{3}(4) \end{aligned}$$

In other words, to get to ***any*** term, we add the common difference to the starting term ***one less*** time than the position of our term.

This same logic may be seen in a geometric sequence such as 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, …

$$\begin{array}{ccccccccccccc} \mathbf{a_1 = 8} & & \mathbf{a_2 = 4} & & \mathbf{a_3 = 2} & & \mathbf{a_4 = 1} & & \mathbf{a_5 = \frac{1}{2}} & & \mathbf{a_6 = \frac{1}{4}} & & \mathbf{a_7 = \frac{1}{8}} \\ & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & \\ & \times\mathbf{\frac{1}{2}} & & \times\mathbf{\frac{1}{2}} & & \times\mathbf{\frac{1}{2}} & & \times\mathbf{\frac{1}{2}} & & \times\mathbf{\frac{1}{2}} & & \times\mathbf{\frac{1}{2}} & \end{array}$$

Thus, in a geometric sequence, rather than add to get to the next term, we multiply. The basic logic is the same. To get to the ***second*** term, we multiply the first term by $\frac{1}{2}$ ***one*** time; to get the the ***third*** term, we multiply the first term by $\frac{1}{2}$ ***two*** times; to get to the ***fourth*** term, we multiply the first term by $\frac{1}{2}$ ***three*** times; and so on.

$$\begin{aligned} a_1 &= 8 \\ a_2 &= 8 \times \frac{1}{2} = 8\left(\frac{1}{2}\right)^{\mathbf{1}} \\ a_3 &= 8 \times \frac{1}{2} \times \frac{1}{2} = 8\left(\frac{1}{2}\right)^{\mathbf{2}} \\ a_4 &= 8 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 8\left(\frac{1}{2}\right)^{\mathbf{3}} \end{aligned}$$

In other words, to get to ***any*** term, we multiply the first term by the common ratio ***one less*** time than the position of our term. We have the following result.

Theorem 7.1. Formulas for Arithmetic and Geometric Sequences

- An arithmetic sequence with the first term a_1 and common difference d is given by

$$a_n = a_1 + (n-1)d, \; n = 1, 2, 3, \ldots$$

- A geometric sequence with the first term a_1 and common ratio r is given by

$$a_n = a_1 r^{n-1}, \; n = 1, 2, 3, \ldots$$

Note: A sequence in the form $a_n = (\text{an expression of } n)$ is arithmetic if the expression is a linear function of n and is geometric if the expression is an exponential function of n, as seen above.

Example 7.1.6. Find an explicit formula for the nth term of the following sequences.

1. $-6, -1, 4, 9, \ldots$

2. $-5, \frac{10}{3}, -\frac{20}{9}, \frac{40}{27}, \ldots$

Solution.

1. Since each term is 5 more than the previous term, this is an arithmetic sequence and the common difference is 5. Noting that the first term is −6, the explicit formula is

$$a_n = -6 + 5(n-1)$$
$$a_n = 5n - 11, \; n = 1, 2, 3, \ldots$$

We can check by evaluating the formula to confirm any term. For example, using the formula, we find $a_3 = 5(3) - 11 = 4$ and this agrees with the third term in the given sequence.

2. There is not a common difference, so we check to see if there is a common ratio. The ratio of the first two terms is

$$\frac{\frac{10}{3}}{-5} = -\frac{10}{15} = -\frac{2}{3}$$

After verifying that $\left(\frac{10}{3}\right)\left(-\frac{2}{3}\right) = -\frac{20}{9}$ and $\left(-\frac{20}{9}\right)\left(-\frac{2}{3}\right) = \frac{40}{27}$, we see that this is a geometric sequence with $r = -\frac{2}{3}$ and $a_1 = -5$. Thus, the explicit formula for the sequence is

$$a_n = -5\left(-\frac{2}{3}\right)^{n-1}, \; n = 1, 2, 3, \ldots$$

□

Example 7.1.7. The first term of an arithmetic sequence is 15 and its 10th term is −12. Find the common difference and an explicit formula.

Solution. We are given that $a_1 = 15$ and $a_{10} = -12$. Using the formula $a_n = a_1 + (n-1)d$ with $n = 10$, we find the common difference as follows:

$$\begin{aligned} a_{10} &= a_1 + (10-1)d \\ -12 &= 15 + 9d \\ 9d &= -27 \\ d &= -3 \end{aligned}$$

The common difference, $d = -3$, and the first term, $a_1 = 15$, give us the formula.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_n &= 15 + (n-1)(-3) \\ a_n &= -3n + 18,\ n = 1,\ 2,\ 3, \ldots \end{aligned}$$

□

Example 7.1.8. The first term of an arithmetic sequence is −12 and its common difference is 5. Which term of this sequence has value 58?

Solution. We are given that $a_1 = -12$ and $d = 5$. We want to find the value of n for which $a_n = 58$. Using the formula from **Theorem 7.1**, we can solve for n.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 58 &= -12 + (n-1)(5) \\ 58 &= 5n - 17 \\ 5n &= 75 \\ n &= 15 \end{aligned}$$

The number 58 is the 15th term of this arithmetic sequence.

□

Example 7.1.9. Find a_{13} in the arithmetic sequence with $a_{10} = -14$ and $a_{30} = -54$.

Solution. Given that $a_{10} = -14$ and $a_{30} = -54$, we would like to find both the first term and the common difference in our search for a_{13}. Using the formula $a_n = a_1 + (n-1)d$, results in the system of equations:

$$\begin{cases} a_{10} = a_1 + (10-1)d \\ a_{30} = a_1 + (30-1)d \end{cases} \Rightarrow \begin{cases} -14 = a_1 + 9d \\ -54 = a_1 + 29d \end{cases}$$

Using elimination, we find $d = -2$ and $a_1 = 4$. So $a_{13} = 4 + (13-1)(-2) = -20$.

□

Example 7.1.10. The fifth term of a geometric sequence is $\frac{4}{5}$ and its common ratio is $\frac{2}{3}$. Find the first term of this sequence. Find an explicit formula.

Solution. Given that $a_5 = \frac{4}{5}$ and $r = \frac{2}{3}$, we use the formula $a_n = a_1 r^{n-1}$ with $n = 5$ to solve for a_1.

$$a_5 = a_1 r^{5-1}$$
$$\frac{4}{5} = a_1 \left(\frac{2}{3}\right)^4$$
$$\frac{4}{5} = \frac{16}{81} a_1$$
$$a_1 = \frac{4}{5} \cdot \frac{81}{16} = \frac{81}{20}$$

The first term is $a_1 = \frac{81}{20}$ and the formula is $a_n = \left(\frac{81}{20}\right)\left(\frac{2}{3}\right)^{n-1}$, $n = 1, 2, 3, \ldots$.

□

7.1 Exercises

1. What are the main differences between using a recursive formula and using an explicit formula to describe an arithmetic sequence?
2. Describe the similarities between exponential functions and geometric sequences. How are they different?

In Exercises 3 – 12, describe how subsequent terms may be determined and state the next two terms in the sequence.

3. $3,\ 5,\ 7,\ 9,\ldots$
4. $\frac{1}{16}, -\frac{1}{8}, \frac{1}{4}, -\frac{1}{2},\ldots$
5. $1,\ \frac{2}{3},\ \frac{4}{5},\ \frac{8}{7},\ldots$
6. $1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{4}{27},\ldots$
7. $1,\ \frac{1}{4},\ \frac{1}{9},\ \frac{1}{16},\ldots$
8. $4,\ 7,\ 12,\ 19,\ 28,\ldots$
9. $-4,\ 2, -10,\ 14, -34,\ldots$
10. $1,\ 1,\ \frac{4}{3},\ 2,\ \frac{16}{5},\ldots$
11. $-\frac{1}{2},\ \frac{1}{4}, -\frac{1}{8},\ \frac{1}{16},\ldots$
12. $1,\ 2,\ 6,\ 24,\ 120,\ldots$

In Exercises 13 – 22, find the first four terms of each sequence. Assume n is a natural number, $n \ge 2$.

13. $a_1 = 9$, $a_n = a_{n-1} + n$
14. $a_1 = 3$, $a_n = (-3)a_{n-1}$
15. $a_1 = -4$, $a_n = \dfrac{a_{n-1} + 2n}{a_{n-1} - 1}$
16. $a_1 = -1$, $a_n = \dfrac{(-3)^{n-1}}{a_{n-1} - 2}$
17. $a_1 = -30$, $a_n = (2 + a_{n-1})\left(\frac{1}{2}\right)^n$
18. $a_1 = 3$, $a_n = a_{n-1} - 1$
19. $a_1 = 12$, $a_n = \dfrac{a_{n-1}}{100}$
20. $a_1 = 2$, $a_n = 3a_{n-1} + 1$
21. $a_1 = -2$, $a_n = \dfrac{a_{n-1}}{(n+1)(n+2)}$
22. $a_1 = 117$, $a_n = \dfrac{1}{a_{n-1}}$

In Exercises 23 - 24, find the first four terms of each sequence. Assume n is a natural number, $n \ge 3$.

23. $a_1 = \frac{1}{24}$, $a_2 = 1$, $a_n = (2a_{n-2})(3a_{n-1})$
24. $a_1 = -1$, $a_2 = 5$, $a_n = a_{n-2}(3 - a_{n-1})$

In Exercises 25 – 36, determine if the sequence is arithmetic, geometric or neither. If it is arithmetic, find the common difference d; if it is geometric, find the common ratio r. Plot the first 5 terms of the sequence.

25. $-6,-12,-24,-48,-96,\ldots$

26. $11.4,\ 9.3,\ 7.2,\ 5.1,\ 3,\ldots$

27. $\frac{1}{3},\ \frac{1}{6},\ \frac{1}{12},\ \frac{1}{24},\ldots$

28. $-1,\ \frac{1}{2},-\frac{1}{4},\ \frac{1}{8},-\frac{1}{16},\ldots$

29. $17,\ 5,-7,-19,\ldots$

30. $4,\ 16,\ 64,\ 256,\ 1024,\ldots$

31. $6,\ 8,\ 11,\ 15,\ 20,\ldots$

32. $2,\ 22,\ 222,\ 2222,\ldots$

33. $0.9,\ 9,\ 90,\ 900,\ldots$

34. $\left\{3n-5\right\}_{n=1}^{\infty}$

35. $a_n = n^2 + 3n + 2$, $n \geq 1$

36. $\left\{3\left(\frac{1}{5}\right)^{n-1}\right\}_{n=1}^{\infty}$

In Exercises 37 – 45, find an explicit formula for the nth term of the sequence.

37. $3,\ 5,\ 7,\ 9,\ldots$

38. $32,\ 24,\ 16,\ 8,\ldots$

39. $-2,-4,-8,-16,\ldots$

40. $1,\ 3,\ 9,\ 27,\ldots$

41. $-5,\ 95,\ 195,\ 295,\ldots$

42. $-17,-217,-417,-617,\ldots$

43. $-1,-\frac{4}{5},-\frac{16}{25},-\frac{64}{125},\ldots$

44. $2,\ \frac{1}{3},\ \frac{1}{18},\ \frac{1}{108},\ldots$

45. $3,-1,\ \frac{1}{3},-\frac{1}{9},\ldots$

In Exercises 46 – 49, use the given information to write the first five terms of the arithmetic sequence.

46. $a_1 = -25$, $d = -9$

47. $a_1 = 0$, $d = \frac{2}{3}$

48. $a_1 = 17$, $a_7 = -31$

49. $a_{13} = -60$, $a_{33} = -160$

In Exercises 50 – 53, use the given information to write the first five terms of the geometric sequence.

50. $a_1 = 8$, $r = 0.3$

51. $a_1 = 5$, $r = \frac{1}{5}$

52. $a_7 = 64$, $a_{10} = 512$

53. $a_6 = 25$, $a_9 = -3.125$

In Exercises 54 – 59, use the given information to find the specified term of the arithmetic sequence.

54. Find a_5 if $a_1 = 3$ and $d = 4$

55. Find a_6 if $a_1 = 6$ and $d = 7$

56. Find a_1 if $a_6 = 12$ and $a_{14} = 28$

57. Find a_1 if $a_7 = 21$ and $a_{15} = 42$

58. Find a_4 if $a_1 = 33$ and $a_7 = -15$

59. Find a_{21} if $a_3 = -17.1$ and $a_{10} = -15.7$

In Exercises 60 – 65, use the given information to find the specified term of the geometric sequence.

60. Find a_5 if $a_1 = 2$ and $r = 3$

61. Find a_4 if $a_1 = 16$ and $r = -\frac{1}{3}$

62. Find a_{12} in the sequence $-1,\ 2, -4,\ 8, \ldots$

63. Find a_7 in the sequence $-2,\ \frac{2}{3}, -\frac{2}{9},\ \frac{2}{27}, \ldots$

64. Find a_8 if $a_1 = 4$ and $a_n = -3a_{n-1}$

65. Find a_{12} if $a_n = -\left(-\frac{1}{3}\right)^{n-1}$

66. Which term of the arithmetic sequence $1, 5, 9, 13, \ldots$ is 185? In other words, if $a_1 = 1$, $a_2 = 5$, and so forth, what is the value of n for which $a_n = 185$?

67. Which term of the arithmetic sequence $47, 44, 41, \ldots$ is -556? In other words, if $a_1 = 47$, $a_2 = 44$, and so forth, what is the value of n for which $a_n = -556$?

68. Which term of the geometric sequence $12, 6, 3, \ldots$ is $\frac{3}{64}$? In other words, if $a_1 = 12$, $a_2 = 6$, and so forth, what is the value of n for which $a_n = \frac{3}{64}$?

69. Find a formula for the general term a_n of the sequence $\left\{-\frac{11}{3}, \frac{13}{9}, -\frac{15}{27}, \frac{17}{81}, -\frac{19}{243}, \ldots\right\}$, assuming the pattern of the first few terms continues, and that the first term is $a_1 = -\frac{11}{3}$.

7.2 Series

Learning Objectives

- Use summation notation.
- Find the sum of a finite arithmetic sequence.
- Solve application problems modeled by arithmetic series.
- Find the value of an infinite geometric series that has a finite sum.
- Find the sum of a finite geometric sequence.
- Solve application problems modeled by geometric series.

As we get older, we benefit or suffer from our accumulated behavior; we may enjoy retirement if we have saved yearly or we may suffer from diseases related to being overweight if we have consumed too many calories daily. If a life is a sequence of events, then the future is influenced by the accumulation of those events. In mathematics, the accumulation or sum of numbers in a sequence is called a **series**. Adding finitely many of those numbers results in a **finite series**, while adding infinitely many of those numbers is an **infinite series**. We express a series, or the sum of a sequence, using the following notation.

Summation Notation

Definition 7.3. Summation Notation: Consider the sequence $\{a_n\}$.

- The sum of the terms a_n, from $n=j$ to $n=p$, can be written as $\sum_{n=j}^{p} a_n = a_j + a_{j+1} + \cdots + a_p$.
- The sum $a_j + a_{j+1} + a_{j+2} + \cdots$ is written as $\sum_{n=j}^{\infty} a_n = a_j + a_{j+1} + a_{j+2} + \cdots$.

The symbol Σ is the Greek capital letter sigma and is read as 'sigma' or 'summation'. The letter n is the **index**. The value j is called the **lower limit** and p is called the **upper limit**.

Note that in the upper limit of the summation $\sum_{n=j}^{p} a_n$ we did not write $n=p$, since n being the index is clear from the lower limit of the summation. It is, however, acceptable to write the upper limit as $n=p$ if you prefer. Also note that we will refer to the value of a series, if it exists, as a sum. The word **sum** may refer to the value of a finite or infinite summation.

Example 7.2.1. Find the following sums.

1. $\sum_{n=2}^{6}(3n-5)$

2. $\sum_{n=1}^{4}\frac{52}{100^n}$

Solution.

1. To evaluate the series $\sum_{n=2}^{6}(3n-5)$, we must find the sum of the sequence generated by the formula $a_n = 3n-5$, where the first term has $n=2$ and the last term has $n=6$. We can do this by finding each term and adding them together.

$$\begin{aligned}\sum_{n=2}^{6}(3n-5) &= \underbrace{(3\cdot 2-5)}_{a_2}+\underbrace{(3\cdot 3-5)}_{a_3}+\underbrace{(3\cdot 4-5)}_{a_4}+\underbrace{(3\cdot 5-5)}_{a_5}+\underbrace{(3\cdot 6-5)}_{a_6}\\ &= 1+4+7+10+13\\ &= 35\end{aligned}$$

2. For $\sum_{n=1}^{4}\frac{52}{100^n}$, we add the sequence generated by $a_n = \frac{52}{100^n}$, starting with $n=1$ and going to $n=4$.

$$\sum_{n=1}^{4}\frac{52}{100^n} = \underbrace{\left(\frac{52}{100^1}\right)}_{a_1}+\underbrace{\left(\frac{52}{100^2}\right)}_{a_2}+\underbrace{\left(\frac{52}{100^3}\right)}_{a_3}+\underbrace{\left(\frac{52}{100^4}\right)}_{a_4} = \frac{52}{100}+\frac{52}{10000}+\frac{52}{1000000}+\frac{52}{100000000}$$

After obtaining a common denominator and simplifying, we have $\frac{52525252}{100000000} = 0.52525252$.

□

Example 7.2.2. Expand the sum $\sum_{k=0}^{4}\frac{(-1)^{k+1}}{k+1}(x-2)^k$.

Solution. Here, we add the terms $a_k = \frac{(-1)^{k+1}}{k+1}(x-2)^k$, starting with $k=0$ and ending with $k=4$:

$$\underbrace{\left(\frac{(-1)^{0+1}}{0+1}(x-2)^0\right)}_{a_0}+\underbrace{\left(\frac{(-1)^{1+1}}{1+1}(x-2)^1\right)}_{a_1}+\underbrace{\left(\frac{(-1)^{2+1}}{2+1}(x-2)^2\right)}_{a_2}+\underbrace{\left(\frac{(-1)^{3+1}}{3+1}(x-2)^3\right)}_{a_3}+\underbrace{\left(\frac{(-1)^{4+1}}{4+1}(x-2)^4\right)}_{a_4}$$

After simplifying, we have $\sum_{k=0}^{4}\frac{(-1)^{k+1}}{k+1}(x-2)^k = -1+\frac{1}{2}(x-2)-\frac{1}{3}(x-2)^2+\frac{1}{4}(x-2)^3-\frac{1}{5}(x-2)^4$

□

In each of the problems in the previous two examples, we were given the formula for a_n along with index values for the first and last terms. Now we will go in the opposite direction. In other words, we start with the terms and generate the summation notation. We make use of our knowledge of arithmetic and geometric sequences, looking for a common difference or common ratio as we go from term to term to determine the behavior of the sequence.

Example 7.2.3. Write the following sums using summation notation.

1. $1+4+9+16+25+36+49$
2. $-5-2+1+\cdots+22$
3. $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$
4. $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots+\frac{5}{1024}$

Solution.

1. Noting that $1=1^2$, $4=2^2$, $9=3^2,\ldots$, $49=7^2$, and starting with $n=1$, then $a_1=1^2$, $a_2=2^2$, $a_3=3^2,\ldots$, $a_7=7^2$. In general, $a_n=n^2$ for $n=1, 2, 3,\ldots,7$. We write the sum of these seven terms using summation notation:

$$1+4+9+16+25+36+49=\sum_{n=1}^{7} n^2$$

2. For $-5-2+1+\cdots+22$, we first determine a general formula for its terms. Since the difference between the first and second term is 3, and the difference between the second and third term is 3, this series has an arithmetic pattern. We still need to find a formula for the terms and the position number for the last term, 22. For the formula, we use the formula for an arithmetic sequence, $a_n=a_1+(n-1)d$, with $a_1=-5$ and $d=3$, to get

$$a_n=-5+(n-1)(3)$$
$$a_n=3n-8,\ n=1, 2, 3,\ldots$$

Now, to determine the value of n for the term 22, we set $a_n=22$ in the formula we just found.

$$a_n=3n-8$$
$$22=3n-8$$
$$n=10$$

In summation notation, $-5-2+1+\cdots+22=\sum_{n=1}^{10}(3n-8)$.

3. Again, we need to determine a pattern for the terms of $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$. Since each term is the previous term multiplied by $-\frac{1}{2}$, the terms follow a geometric pattern. Using the formula for a geometric sequence, $a_n=a_1r^{n-1}$, with $a_1=5$ and $r=-\frac{1}{2}$, we find $a_n=5\left(-\frac{1}{2}\right)^{n-1}$, $n=1, 2, 3,\ldots$

The lack of a last term, indicated by '… ', indicates we are adding infinitely many terms. Finally, using summation notation,

$$5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots=\sum_{n=1}^{\infty}5\left(-\frac{1}{2}\right)^{n-1}$$

4. The terms in the series $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots+\frac{5}{1024}$ are the same as the terms in the series in the previous part of this example. However, this time there is a last term, $\frac{5}{1024}$. As before, the sum can be written as $\sum_{n=1}^{?}5\left(-\frac{1}{2}\right)^{n-1}$, so we need only to determine the value of n for which $\frac{5}{1024}=5\left(-\frac{1}{2}\right)^{n-1}$. Since $\frac{5}{1024}=5\left(-\frac{1}{2}\right)^{10}$, $n=11$, and

$$5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots+\frac{5}{1024}=\sum_{n=1}^{11}5\left(-\frac{1}{2}\right)^{n-1}$$

□

Although the summation notation is new, it is just a way of writing addition for many terms that are each generated in the same way. Because it is just addition, the usual properties of addition hold. For example, the order of addition does not matter. Below, we list properties of summation, using mathematical notation and noting that p may represent an integer or infinity. In the case where $p=\infty$, these properties hold if each infinite sum is defined. We will discuss certain infinite sums later. They will be discussed extensively in Calculus.

Properties of Summation

- $\sum_{n=j}^{p}(a_n \pm b_n)=\sum_{n=j}^{p}a_n \pm \sum_{n=j}^{p}b_n$
- $\sum_{n=j}^{p}a_n=\sum_{n=j}^{h}a_n+\sum_{n=h+1}^{p}a_n$, for any integer h with $j\le h<p$
- $\sum_{n=j}^{p}c\,a_n=c\sum_{n=j}^{p}a_n$, for any constant c
- $\sum_{n=j}^{p}a_n=\sum_{n=j+h}^{p+h}a_{n-h}$, for any integer h (if $p=\infty$, replace $p+h$ with ∞)

We now turn our attention to finding sums, limiting our sums to arithmetic and geometric series. We begin with arithmetic series.

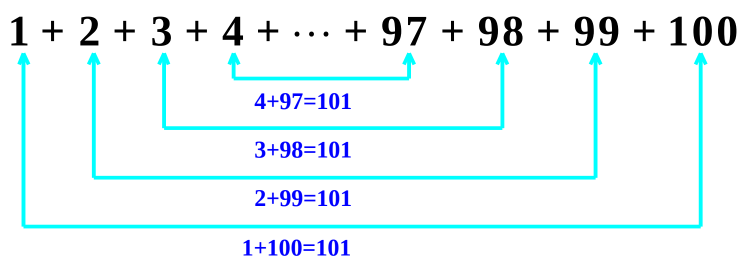
Arithmetic Series

There is a story often told in mathematics classes about Carl Friedrich Gauss (1777–1855), one of the world's most famous mathematicians, when he was about 9 years old. The legend goes like this: Gauss's teacher often gave Gauss and his classmates tedious arithmetic tasks. On one such occasion, the teacher asked students to add all of the integers from 1 to 100:

$$1+2+3+4+\cdots+97+98+99+100$$

All of the students, except Gauss, added the numbers one at a time. When the teacher asked him why he was not working on the problem, Gauss replied that he already knew the answer, 5050. The teacher was astonished and asked how he knew the answer without doing the computation. Gauss explained that the sum of the first term and the last term is 101; the sum of the second term and the second to the last term is 101; and the sum of the third and the third to the last term is, again, 101. As a matter of fact, you always get 101 if you continue in this manner. There are a total of 50 sums of 101, so the sum of all integers from 1 to 100 is $50(101)=5050$.

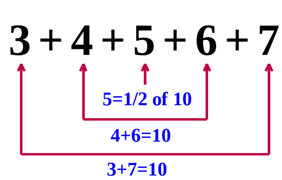
Figure 7.2. 1



In other words, we are taking the first term plus the last term and then multiplying that sum by half the number of terms in the series. Noting that an **arithmetic sum** is a sum of consecutive terms of an arithmetic sequence, if we denote the sum of the first n terms of an arithmetic sequence $\{a_k\}$ by S_n, then the above discussion implies that $S_n=\sum_{k=1}^{n} a_k = a_1+a_2+\cdots+a_n=\frac{n}{2}(a_1+a_n)$.

Before stating a theorem, let's quickly check that this formula also works for an odd number of terms, say for $3+4+5+6+7$.

Figure 7.2. 2



We see that $\frac{n}{2}(a_1+a_n)=\frac{5}{2}(3+7)=25$ is the correct sum. The formula $\frac{n}{2}(a_1+a_n)$ does work for an odd number of terms since the middle term is half the sum of the first and last terms.

Theorem 7.2. Arithmetic Sum:

Consider the arithmetic sequence $a_k=a_1+(k-1)d$, $k=1, 2, 3, \ldots$, with the first term a_1 and the common difference d.

The sum of the first n terms, $S_n=\sum_{k=1}^{n}a_k=a_1+a_2+\cdots+a_n=a_1+(a_1+d)+\cdots+(a_1+(n-1)d)$, is

$$S_n=\frac{n}{2}(a_1+a_n)=\frac{n}{2}(2a_1+(n-1)d).$$

We can prove this theorem as follows. Consider the finite arithmetic sequence $a_k=a_1+(k-1)d$,

$k=1, 2, \ldots, n$ and its sum $S_n=\sum_{k=1}^{n}\left[a_1+(k-1)d\right]=a_1+(a_1+d)+\cdots+(a_1+(n-2)d)+(a_1+(n-1)d)$.

To determine the value of the sum, we write the sum twice, the second time in reverse order, and add the two.

$$\begin{array}{rccccc}
S_n= & a_1+ & (a_1+d) & +\cdots+ & (a_1+(n-2)d)+ & (a_1+(n-1)d) \\
S_n= & (a_1+(n-1)d)+ & (a_1+(n-2)d) & +\cdots+ & (a_1+d)+ & a_1 \\
2S_n= & (2a_1+(n-1)d)+ & (2a_1+(n-1)d) & +\cdots+ & (2a_1+(n-1)d)+ & (2a_1+(n-1)d)
\end{array}$$

The right side is the sum of n copies of $(2a_1+(n-1)d)$, so

$$2S_n=n(2a_1+(n-1)d)$$
$$S_n=\frac{n}{2}(2a_1+(n-1)d)$$

Using the fact that the last, or nth, term in the sequence is $a_n=a_1+(n-1)d$, we can also write this sum as $S_n=\frac{n}{2}(a_1+a_n)$.

Example 7.2.4. Find the value of the arithmetic[3] sum $\sum_{k=1}^{12}(4k-5)$.

Solution. We could write out each of the terms and then add them, but this would be tedious. Instead, we use the formula. There are 12 terms, so $n=12$. We find the value of the first and the last term by setting $k=1$ and $k=12$, respectively.

$$k=1: \quad a_1=4(1)-5=-1$$
$$k=12: \quad a_{12}=4(12)-5=43$$

[3] To verify that this sum is arithmetic, we find that we have a common difference, i.e. $a_{k+1}-a_k=4$.

The sum is $S_{12} = \frac{12}{2}(-1+43) = 252$.

□

Example 7.2.5. The first term of an arithmetic sequence is 15 and its 10th term is −12. Find the sum of its first 10 terms. Find the sum of its first 20 terms.

Solution. We are given $a_1 = 15$ and $a_{10} = -12$. The sum of the first 10 terms is

$$\begin{aligned} S_{10} &= \frac{10}{2}\left(15+(-12)\right) \\ &= 15 \end{aligned}$$

To find the sum of the first 20 terms, we need to find the value of d. Using $a_n = a_1 + (n-1)d$ with $a_1 = 15$, $a_{10} = -12$, and $n = 10$, we have

$$\begin{aligned} -12 &= 15 + (10-1)d \\ -27 &= 9d \\ d &= -3 \end{aligned}$$

Then, using the formula $S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$,

$$\begin{aligned} S_{20} &= \frac{20}{2}\left(2(15)+(20-1)(-3)\right) \\ &= 10(30-57) \\ &= -270 \end{aligned}$$

□

Example 7.2.6. The first term of an arithmetic sequence is -12 and its common difference is 5. How many consecutive terms, starting with the first term, of this sequence must be added to get a sum of 345?

Solution. Given $a_1 = -12$ and $d = 5$, we are looking for n so that $S_n = 345$ where $S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$:

$$\begin{aligned} \frac{n}{2}\left[2(-12)+(n-1)(5)\right] &= 345 \\ n(-24+5n-5) &= 690 \\ 5n^2 - 29n - 690 &= 0 \end{aligned}$$

We solve the resulting equation for n, with the assistance of the Quadratic Formula.

$$n = \frac{29 \pm \sqrt{(-29)^2 - 4(5)(-690)}}{2(5)}$$
$$n = \frac{29 \pm \sqrt{14641}}{10}$$
$$n = \frac{29 \pm 121}{10}$$

Now, $n = -9.2$ or $n = 15$. Since n must be a whole number, we find $n = 15$ and conclude that adding the first 15 terms of the sequence will result in 345.

□

Applications of Arithmetic Series

Example 7.2.7. On the Sunday after a minor surgery, Margaret is able to walk a half-mile. Each Sunday, she walks an additional quarter mile. After 8 weeks, what will be the total number of miles she has walked?

Solution. This problem can be modeled as a sum of an arithmetic sequence with $a_1 = \frac{1}{2}$ and $d = \frac{1}{4}$.

We are looking for the total number of miles walked after 8 weeks, so we know that $n = 8$ and that we are looking for S_8. Using the formula $S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$, we have

$$\begin{aligned} S_8 &= \frac{8}{2}\left(2\left(\frac{1}{2}\right) + (8-1)\left(\frac{1}{4}\right)\right) \\ &= 4\left(1 + \frac{7}{4}\right) \\ &= 11 \end{aligned}$$

Margaret walked a total of 11 miles.

□

Before moving on to geometric series, we ponder the results of summing infinitely many arithmetic terms. Suppose Gauss had been asked to add up all of the natural numbers.

$$1 + 2 + 3 + 4 + 5 + \cdots$$

This would mean adding progressively larger and larger numbers or, in other words, bigger and bigger terms. The sum just keeps getting larger. This sum will not be a finite number and is said to 'go to infinity' or **diverge**. This always happens for arithmetic sequences where $a_1 \neq 0$, even if $d = 0$. For example, if $a_1 = 1$ and $d = 0$, the sum $1 + 1 + 1 + \cdots$ is not a finite number. We leave it to the reader to verify that a non-zero infinite arithmetic sum will not have a finite value; that is, it diverges.

Geometric Series

We begin by considering the geometric sequence $a_k = a_1 r^{k-1}$. The geometric sum

$S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$ is the sum of the first n terms of this sequence. To find the value of S_n, we use a bit of arithmetic; we multiply the terms of the series by r, subtract the result from S_n to give us $S_n - r S_n$, then proceed to solve for S_n.

$$
\begin{array}{rcccccccl}
S_n = & a_1 + & a_1 r + & a_1 r^2 + & \cdots & + a_1 r^{n-2} + & a_1 r^{n-1} & & \\
r S_n = & & a_1 r + & a_1 r^2 + & \cdots & + a_1 r^{n-2} + & a_1 r^{n-1} + & a_1 r^n & \\
\hline
S_n - r S_n = & a_1 & & & & & & -a_1 r^n &
\end{array}
$$

Thus,

$$
\begin{aligned}
S_n - r S_n &= a_1 - a_1 r^n \\
S_n (1 - r) &= a_1 - a_1 r^n \\
S_n &= \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1 (1 - r^n)}{1 - r}
\end{aligned}
$$

Theorem 7.3. Geometric Sum (finite number of terms):

Consider the geometric sequence $a_k = a_1 r^{k-1}$ with the first term a_1 and the common ratio r, $r \neq 1$.

The sum of the first n terms, $S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 + a_1 r + \cdots + a_1 r^{n-1}$, is

$$S_n = \frac{a_1 (1 - r^n)}{1 - r}$$

In the case when $r = 1$, $S_n = \underbrace{a_1 + a_1 + a_1 + \cdots + a_1}_{n \text{ terms}} = n a_1$.

Example 7.2.8. Find the value of the sum $5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \cdots + \frac{5}{1024}$.

Solution. This is a finite geometric sum with $a_1 = 5$ and $r = -\frac{1}{2}$, since each term in the sum is the previous term multiplied by $-\frac{1}{2}$. We need only to know the number of terms, n, to find the sum with the formula in **Theorem 7.3**. To determine the value of n, we use the formula $a_n = a_1 r^{n-1}$ with $a_1 = 5$ and $a_n = \frac{5}{1024}$.

$$\frac{5}{1024} = 5\left(-\frac{1}{2}\right)^{n-1}$$
$$\frac{1}{1024} = \left(-\frac{1}{2}\right)^{n-1}$$
$$\left(-\frac{1}{2}\right)^{10} = \left(-\frac{1}{2}\right)^{n-1} \quad \text{since } \left(-\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

We have $n-1=10 \Rightarrow n=11$. We may now use the formula $S_n = \dfrac{a_1\left(1-r^n\right)}{1-r}$ with $n=11$, $a_1=5$, and $r=-\dfrac{1}{2}$ to find the sum.

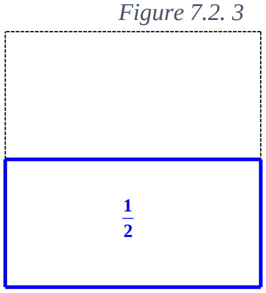
$$S_{11} = \frac{5\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{5\left(1+\frac{1}{2048}\right)}{\frac{3}{2}} = \frac{10245}{2048}\cdot\frac{2}{3} = \frac{3415}{1024}$$

□

An interesting question, to be explored further in Calculus, is 'when is it possible to add an infinite number of numbers and get a finite sum?' We talked about adding an infinite number of terms in an arithmetic sequence. Now we examine the sum of infinitely many geometric terms. Suppose you add all of the terms starting with $\frac{1}{2}$ where each subsequent term is half of the previous term: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$.

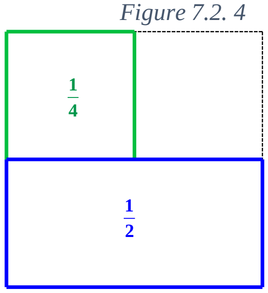
As an illustration, in the following diagram the indicated square has a side length of 1, and we sum areas within the square as we move from left to right.

Figure 7.2. 3



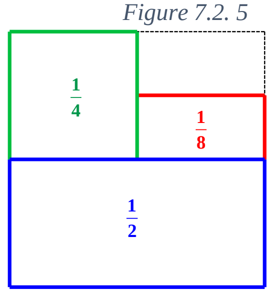
$\frac{1}{2}$

Figure 7.2. 4



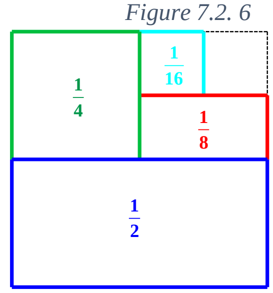
$\frac{1}{2}+\frac{1}{4}$

Figure 7.2. 5



$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

Figure 7.2. 6



$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

Notice that we keep adding smaller and smaller pieces, filling in the square of area 1. So, from the left to the right, the sum is getting closer to 1, and indeed the sum is 1.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

While this geometric series has a finite value, this is obviously not the case for all geometric series. For example, the geometric series $1+2+4+8+\cdots$ diverges, much like the arithmetic series we discussed earlier. As it turns out, the series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ has a finite value when $-1<r<1$; in this case we say the series **converges**. For all other values of r, the infinite geometric series will not have a finite value. As to what happens and why, we leave that discussion to your Calculus course.

Theorem 7.4. Geometric Sum (infinite number of terms):

Consider the geometric sequence $a_k = a_1 r^{k-1}$ with first term a_1 and common ratio r, $-1<r<1$. Then the infinite geometric series $S=\sum_{k=1}^{\infty} a_1 r^{k-1} = a_1 + a_1 r + a_1 r^2 + \cdots$ is the sum of all terms of this infinite geometric sequence where

$$S = \frac{a_1}{1-r}$$

If $|r| \geq 1$, the sum $\sum_{k=1}^{\infty} a_1 r^{k-1}$, with $a_1 \neq 0$, is not defined.

Example 7.2.9. Find the value of the series, if it exists.

1. $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$
2. $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)^{k-1}$
3. $\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{k-1}$

Solution.

1. The series $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$ is geometric with $a_1 = 5$ and $r = -\frac{1}{2}$. Because $|r|<1$, the series will converge to a value. We use the formula in **Theorem 7.4** to find that value.

$$5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots = \frac{5}{1-\left(-\frac{1}{2}\right)} = \frac{5}{\left(\frac{3}{2}\right)} = \frac{10}{3}$$

2. The series $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)^{k-1}$ is written in summation notation. Since $\frac{3}{2}$ is the number being raised to a power, this means we are multiplying each term by $\frac{3}{2}$ to get to the next term. Thus, the ratio is $\frac{3}{2}$, which is larger than the absolute value of 1, so the series will diverge.

3. Again, the series is written in summation notation. However, for $\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{k-1}$, we have $r=\frac{2}{3}$, which has an absolute value less than 1, so the series will converge to a value. We find $a_1=\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{1-1}=\frac{3}{2}$ and use the formula from **Theorem 7.4** to find the sum.

$$\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{k-1}=\frac{\frac{3}{2}}{1-\frac{2}{3}}=\left(\frac{3}{2}\right)\cdot\left(\frac{3}{1}\right)=\frac{9}{2}$$

□

Applications of Geometric Series

A rational number is a real-valued number that is the ratio of two integers or, in decimal notation, has a decimal expansion that ends or repeats. The following application of geometric series allows us to write a repeating decimal as a ratio of two integers.

Example 7.2.10. Write the rational number $4.\overline{52}=4.525252\ldots$ as a ratio of two integers.

Solution. We begin by noting that the decimal portion of this number can be written as a geometric series, to which we may apply the geometric series formula.

$$\begin{aligned}4.\overline{52}&=4.525252\ldots\\&=4+0.525252\ldots\\&=4+\underbrace{0.52+0.0052+0.000052+\cdots}_{\text{geometric series}}\quad a_1=0.52=\frac{52}{100}\text{ and } r=0.01=\frac{1}{100}\\&=4+\frac{\frac{52}{100}}{1-\frac{1}{100}}\end{aligned}$$

We find $4.\overline{52}=4+\frac{52}{99}=\frac{448}{99}$.

□

An interesting fact that can be shown with the method used in **Example 7.2.10** is that $0.\overline{9}=0.999\cdots=1$. Try it on your own!

An important applicaton of the geometric sum formula is the investment plan called an **annuity**. Annuities differ from the kind of investments we studied in **Section 4.5** in that payments are deposited into the account on an on-going basis. In the following example, we look at an **ordinary annuity**, which requires equal payments made at the end of consecutive periods, each having the same length. To find the value of an annuity, we must sum all of the payments and the interest earned.

Example 7.2.11. A deposit of \$50 is made at the end of each month in a savings account that offers 6% annual interest, compounded monthly. Find the value of the savings account after 30 years.

Solution. We begin by finding the values, at the end of 30 years, of the individual monthly deposits of \$50. In **Section 4.5**, we used the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ for principal P, annual interest rate r, n compoundings per year, and t years. For simplicity, here, we let $m = nt$ be the number of compounding periods in t years. In this example, we have $r = 0.06$ and $P = 50$, so $A = 50\left(1+\frac{0.06}{12}\right)^{m} = 50(1.005)^{m}$. The initial deposit of \$50 will be compounded over $m = (12)(30) - 1 = 359$ months; the second deposit of \$50 will be compounded over $m = 358$ months, the third over $m = 357$ months, etc.

Deposit #	Months Compounding	Resulting Value
1	359	$50(1.005)^{359}$
2	358	$50(1.005)^{358}$
3	357	$50(1.005)^{357}$
$\vdots$	$\vdots$	$\vdots$
360	0	$50(1.005)^{0}$

The sum of these values that result from monthly deposits is

$$\begin{aligned} &50(1.005)^{359} + 50(1.005)^{358} + 50(1.005)^{357} + \cdots + 50(1.005)^{0} \\ &= 50(1.005)^{0} + 50(1.005)^{1} + 50(1.005)^{2} + \cdots + 50(1.005)^{359} \\ &= 50 + 50(1.005)^{1} + 50(1.005)^{2} + \cdots + 50(1.005)^{359} \end{aligned}$$

This is the sum of the first 360 terms of the geometric sequence $\left\{a_1 r^{k-1}\right\}_{k=1}^{\infty}$ with $a_1 = 50$ and $r = 1.005$. Its value is

$$\begin{aligned} S_{360} &= \frac{50\left(1-1.005^{360}\right)}{1-1.005} \quad \text{using the formula } S_m = \frac{a_1\left(1-r^m\right)}{1-r} \\ &\approx 50225.75 \end{aligned}$$

Thus, the account contains \$50,225.75 after 30 years.

□

7.2 Exercises

1. What is the difference between an arithmetic sequence and an arithmetic series?

2. Describe the criteria for determining if an infinite geometric series has a finite sum. Give an example of an infinite geometric series that has a finite sum, and another that does not.

In Exercises 3 – 11, find the value of the sum.

3. $\sum_{a=1}^{14} a$

4. $\sum_{n=1}^{6} n(n-2)$

5. $\sum_{k=1}^{17} k^2$

6. $\sum_{g=4}^{9} (5g+3)$

7. $\sum_{k=3}^{8} \frac{1}{k}$

8. $\sum_{j=0}^{5} 2^j$

9. $\sum_{k=0}^{2} (3k-5)x^k$

10. $\sum_{i=1}^{4} \frac{1}{4}(i^2+1)$

11. $\sum_{n=1}^{100} (-1)^n$

In Exercises 12 – 21, rewrite the sum using summation notation.

12. $8+11+14+17+20$

13. $1-2+3-4+5-6+7-8$

14. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$

15. $1+2+4+\cdots+2^{29}$

16. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}$

17. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{179\cdot 180}$

18. $7+4+1-2-\cdots-83$

19. $-\ln(3)+\ln(4)-\ln(5)+\cdots+\ln(20)$

20. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}$

21. $\frac{1}{2}(x-5)+\frac{1}{4}(x-5)^2+\frac{1}{6}(x-5)^3+\frac{1}{8}(x-5)^4$

In Exercises 22 – 41, use formulas from this section to find the sum, if possible.

22. $\sum_{n=1}^{10} (5n+3)$

23. $\sum_{n=1}^{20} (2n-1)$

24. $\sum_{k=0}^{15} (3-k)$

25. $\sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$

26. $\sum_{n=1}^{5} \left(\frac{3}{2}\right)^n$

27. $\sum_{k=0}^{5} 2\left(\frac{1}{4}\right)^k$

28. $1+4+7+\cdots+295$

29. $4+2+0-2-\cdots-146$

30. $1+3+9+\cdots+2187$

31. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{256}$

32. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\cdots+\frac{3}{256}$

33. $4+2+1+\frac{1}{2}+\cdots$

34. $\frac{1}{2}+1+2+4+\cdots$

35. $-1-\frac{1}{4}-\frac{1}{16}-\frac{1}{64}-\cdots$

36. $\sum_{k=1}^{\infty} 3\left(\frac{1}{4}\right)^{k-1}$

37. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n}$

38. $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\cdots$

39. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^{n-1}$

40. $\sum_{n=1}^{\infty} 8\left(\frac{4}{5}\right)^{n-1}$

41. $\sum_{j=1}^{\infty} \frac{3}{7}(4)^{j-1}$

42. The sum of terms $50-k^2$ from $k=x$ through $k=7$ is 115. What is x?

43. Write an explicit formula for a_k such that $\sum_{k=0}^{6} a_k = 189$. Assume this is an arithmetic series.

44. Find the smallest value of n such that $\sum_{k=1}^{n} (3k-5) > 100$.

45. How many terms must be added before the series $-1-3-5-7-\cdots$ has a sum less than −75?

46. Piotr devised a week-long study plan to prepare for finals. On the first day, he plans to study for 1 hour, and each successive day he will increase his study time by 30 minutes. How many hours will Piotr have studied after one week?

47. A testing center is designed with 10 seats in the first row, 12 seats in the second row, 14 seats in the third row, and so forth. The testing center has 15 rows of seating. What is the maximum number of students who may be testing at any one time?

48. A brick wall is built with 300 bricks in the first row, 299 bricks in the second row, and each successive row contains one less brick. If the top row contains 177 bricks, what is the total number of bricks required to build the wall?

49. Find the sum $1+2+3+\cdots+1000$.

In Exercises 50 – 55, express the repeating decimal as a fraction of integers.

50. $0.\overline{7}$

51. $0.\overline{13}$

52. $2.\overline{3}$

53. $4.\overline{17}$

54. $10.\overline{159}$

55. $-5.8\overline{67}$

In Exercises 56 – 61, compute the future value of the annuity with the given terms. In all cases, assume the payment is made at the end of each month, the interest rate given is the annual rate, and interest is compounded at the end of each month.

56. payments are \$300, interest rate is 2.5%, term is 17 years.

57. payments are \$50, interest rate is 1.0%, term is 30 years.

58. payments are \$100, interest rate is 2.0%, term is 20 years.

59. payments are \$100, interest rate is 2.0%, term is 25 years.

60. payments are \$100, interest rate is 2.0%, term is 30 years.

61. payments are \$100, interest rate is 2.0%, term is 35 years.

62. Discuss with your classmates what goes wrong when trying to find the following sums.[4]

(a) $\sum_{k=1}^{\infty} 2^{k-1}$ (b) $\sum_{k=1}^{\infty} (1.0001)^{k-1}$ (c) $\sum_{k=1}^{\infty} (-1)^{k-1}$

[4] When in doubt, write them out!

7.3 Binomial Expansion

Learning Objectives

- Expand binomial powers using
  - Pascal's Triangle
  - Binomial Theorem
- Find an indicated term in the expansion of a binomial power.

A binomial is simply a polynomial with two terms. In this section, we are interested in binomial powers of the form $(a+b)^n$, for $n=0,\ 1,\ 2,\ 3,\ldots$. The expressions $(a+b)^2$ and $(a+b)^3$ occur frequently in simplifying, factoring, and solving equations. They can be expanded using the following identities:

Identity	In sentence form
$(a+b)^2 = a^2+2ab+b^2$	*The sum of two terms squared is the first squared, plus twice the first times the second, plus the second squared.*
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$	*The sum of two terms cubed is the first cubed, plus three times the first squared times the second, plus three times the first times the second squared, plus the second cubed.*

These identities can be applied for even more complicated binomials, as demonstrated below.

$$\begin{aligned}(2x+3y)^2 &= ((2x)+(3y))^2\\ &= (2x)^2+2(2x)(3y)+(3y)^2\\ &= 4x^2+12xy+9y^2\end{aligned}$$

$$\begin{aligned}(a-b)^3 &= (a+(-b))^3\\ &= a^3+3a^2(-b)+3a(-b)^2+(-b)^3\\ &= a^3-3a^2b+3ab^2-b^3\end{aligned}$$

Example 7.3.1. Simplify $4(a-3)^3-a(2a-9)^2$.

Solution.

$$\begin{aligned}4(a-3)^3-a(2a-9)^2 &= 4\left(a^3+3a^2(-3)+3a(-3)^2+(-3)^3\right)-a\left((2a)^2+2(2a)(-9)+(-9)^2\right)\\ &= 4\left(a^3-9a^2+27a-27\right)-a\left(4a^2-36a+81\right)\\ &= 4a^3-36a^2+108a-108-4a^3+36a^2-81a\\ &= 27a-108\end{aligned}$$

□

Example 7.3.2. Find all real solutions to $x^3+6x^2+12x+8=125$.

Solution. The left side of this equation is the expansion of $(a+b)^3$ with $a=x$ and $b=2$.

$$x^3+6x^2+12x+8=125$$
$$x^3+3(x^2)(2)+3(x)(2^2)+2^3=125$$
$$(x+2)^3=5^3$$

Now, $x+2=5$, from which we find $x=3$.

□

Consider $(a+b)^n$ in general. This is the product of n copies of $(a+b)$, which we can expand by multiplying out term by term.

$$(a+b)^n=\overbrace{(a+b)\times(a+b)\times\cdots\times(a+b)}^{n\text{ copies}}$$
$$=a^n+\square a^{n-1}b+\square a^{n-2}b^2+\cdots+\square a^2b^{n-2}+\square ab^{n-1}+b^n$$

Multiplying out the a terms in each of the n copies, we get a^n, and there is only one way of obtaining it, so the first term of the expansion is a^n. Now, multiplying the a terms from the $n-1$ copies and the b from the remaining copy, we get $a^{n-1}b$. However, we can obtain this term more than one way. Reducing the number of a terms and replacing them with b terms, we get the expression $a^{n-k}b^k$ for $k=2,\ 3,\ 4,\cdots,n-1$. Again, we can obtain each of these in more than one way. Finally, multiplying the b terms from each copy, we get b^n, and there is only one way of obtaining it, so the last term of the expansion is b^n.

Pascal's Triangle

Except for the first and last terms, at this time, we don't know the coefficients of the other terms. In order to discover these, we list expansions of several binomial powers and try to recognize the pattern of the terms and coefficients.

$$(a+b)^0=1$$
$$(a+b)^1=a+b$$
$$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$$
$$(a+b)^3=(a+b)(a+b)^2=a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4=(a+b)(a+b)^3=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Here, the expansion is written in descending order of the powers of the first term. As discussed above, in the expansion of $(a+b)^n$ for $n=1,\ 2,\ 3,\ldots$, the first term is a^n. For the middle terms we decrease the power of a by one and increase the power of b by one until we reach the last term b^n. To discover what is going on with the middle terms, look at the coefficients.

$$\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1}
\end{array}$$

Notice that each new row can be formed from the previous one by adding a 1 at the beginning and another 1 at the end. Then each term in the middle is the sum of the two above it, as shown below.

3rd row, coefficients of expansion of $(a+b)^2$:

$$\begin{array}{ccccc}
 & & \mathbf{1} & & \\
 & \mathbf{1}\searrow & & \swarrow\mathbf{1} & \\
\boxed{\mathbf{1}} & & \underline{\mathbf{1+1}} & & \boxed{\mathbf{1}}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccc}
 & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{2} & & \mathbf{1}
\end{array}$$

4th row, coefficients of expansion of $(a+b)^3$:

$$\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{1} & & \\
 & \mathbf{1}\searrow & & \swarrow\mathbf{2}\searrow & & \swarrow\mathbf{1} & \\
\boxed{\mathbf{1}} & & \underline{\mathbf{1+2}} & & \underline{\mathbf{2+1}} & & \boxed{\mathbf{1}}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1}
\end{array}$$

5th row, coefficients of expansion of $(a+b)^4$:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & \\
 & \mathbf{1}\searrow & & \swarrow\mathbf{3}\searrow & & \swarrow\mathbf{3}\searrow & & \swarrow\mathbf{1} & \\
\boxed{\mathbf{1}} & & \underline{\mathbf{1+3}} & & \underline{\mathbf{3+3}} & & \underline{\mathbf{3+1}} & & \boxed{\mathbf{1}}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1}
\end{array}$$

Following this pattern, using the last row, we get a new row.

$$\begin{array}{ccccccccccc}
 & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1}
\end{array}$$

This row is precisely the coefficients of the expansion $(a+b)^5$.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Of course, you can check this by performing the operation $(a+b)^5 = (a+b)(a+b)^4$.

The table formed by these rows of coefficients is called **Pascal's Triangle**. Blaise Pascal was a French mathematician and physicist who lived in the 17th century. However, this result was well known both in Iran and China in the 10th century and might even go as far back as the 2nd century BCE in India. Notice that there is symmetry in each row, making it easy to check our work.

Example 7.3.3. Use Pascal's Triangle to expand the following.

1. $(2x-3)^4$ 2. $(a-b)^5$ 3. $(a+b)^6$

Solution. We write one more row of Pascal's Triangle for $(a+b)^n$.

$n=0$							**1**						
$n=1$						**1**		**1**					
$n=2$					**1**		**2**		**1**				
$n=3$				**1**		**3**		**3**		**1**			
$n=4$			**1**		**4**		**6**		**4**		**1**		
$n=5$		**1**		**5**		**10**		**10**		**5**		**1**	
$n=6$	**1**		**6**		**15**		**20**		**15**		**6**		**1**

1. For $(2x-3)^4$, we have $n=4$.

$$\begin{aligned}(2x-3)^4 &= ((2x)+(-3))^4\\ &= (2x)^4+4(2x)^3(-3)+6(2x)^2(-3)^2+4(2x)(-3)^3+(-3)^4\\ &= 16x^4-96x^3+216x^2-216x+81\end{aligned}$$

2. We use the row for $n=5$ to expand $(a-b)^5$.

$$\begin{aligned}(a-b)^5 &= (a+(-b))^5\\ &= a^5+5a^4(-b)+10a^3(-b)^2+10a^2(-b)^3+5a(-b)^4+(-b)^5\\ &= a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5\end{aligned}$$

3. The row for $n=6$ gives us the coefficients for the expansion of $(a+b)^6$.

$$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$

□

The Binomial Theorem

Recall our discussion regarding the 'missing coefficients' in the expansion of $(a+b)^n$.

$$\begin{aligned}(a+b)^n &= \overbrace{(a+b)\times(a+b)\times\cdots\times(a+b)}^{n\text{ copies}}\\ &= a^n+\square a^{n-1}b+\square a^{n-2}b^2+\cdots+\square a^2b^{n-2}+\square ab^{n-1}+b^n\end{aligned}$$

Pascal's Triangle is a recursive method to find these coefficients. The Binomial Theorem provides us with a way of filling in these 'missing coefficents' explicitly. There may be occasions when we require only one term, say the fifth term, in an expansion and the Binomial Theorem allows us to find that single term. Before proceeding, we introduce the **factorial**, and note that the notation for the factorial is $n!$.

Definition 7.4. Factorial: The factorial of a nonnegative integer is defined as follows.

$$0! = 1$$
$$n! = 1 \times 2 \times 3 \times \cdots \times n \quad \text{for } n = 1,\ 2,\ 3, \ldots$$

Example 7.3.4. Evaluate $n!$ for $n = 0,\ 1,\ 2, \ldots,\ 5$.

Solution. Using **Definition 7.4**, we have the following.

$$\begin{aligned}
0! &= 1 \\
1! &= 1 \\
2! &= 1 \times 2 = 2 \\
3! &= 1 \times 2 \times 3 = 6 \\
4! &= 1 \times 2 \times 3 \times 4 = 24 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 = 120
\end{aligned}$$

□

Note that $0! = 1$ does not follow the pattern of the definition $n! = 1 \times 2 \times 3 \times \cdots \times n$, and also note that $1! = 1$. In the next example, we evaluate some expressions involving factorials.

Example 7.3.5. Evaluate the following.

1. $(9-3)!$
2. $\dfrac{5!}{2!}$
3. $\dfrac{10!}{8!}$
4. $\dfrac{100!}{97!3!}$

Solution.

1. $$\begin{aligned}(9-3)! &= 6! \\ &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \\ &= 720\end{aligned}$$

2. $$\begin{aligned}\frac{5!}{2!} &= \frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times 2} \\ &= 3 \times 4 \times 5 \\ &= 60\end{aligned}$$

3. $$\begin{aligned}\frac{10!}{8!} &= \frac{\overbrace{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}^{8!} \times 9 \times 10}{8!} \\ &= 9 \times 10 \\ &= 90\end{aligned}$$

4. $$\begin{aligned}\frac{100!}{97!3!} &= \frac{\overbrace{1 \times 2 \times 3 \times \cdots \times 97}^{97!} \times 98 \times 99 \times 100}{(97!)(1 \times 2 \times 3)} \\ &= \frac{98 \times 99 \times 100}{1 \times 2 \times 3} \\ &= 161{,}700\end{aligned}$$

□

Note: Although many calculators have a button for factorials, a calculator cannot evaluate $100!$. So, to evaluate part 4 of the previous example, it is essential to recognize that $100! = (97!)\times 98\times 99\times 100$ and simplify the fraction.

Let us take another look at the earlier expansion $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ and, as we work toward the Binomial Theorem, rewrite the coefficients in terms of factorials. Our goal is that each coefficient is a fraction with a numerator of $5!$.

$$\begin{aligned}(a+b)^5 &= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5\\ &= \frac{5!}{5!}a^5 + \frac{1\times2\times3\times4\times5}{1\times2\times3\times4\times1}a^4b + \frac{1\times2\times3\times4\times5}{1\times2\times3\times2}a^3b^2 + \frac{1\times2\times3\times4\times5}{1\times2\times2\times3}a^2b^3 + \frac{1\times2\times3\times4\times5}{1\times2\times3\times4}ab^4 + \frac{5!}{5!}b^5\\ &= \frac{5!}{5!}a^5 + \frac{5!}{4!1!}a^4b + \frac{5!}{3!2!}a^3b^2 + \frac{5!}{2!3!}a^2b^3 + \frac{5!}{4!}ab^4 + \frac{5!}{5!}b^5\end{aligned}$$

Seeing a pattern beginning to emerge, we work with the equation a bit more to get

$$(a+b)^5 = \frac{5!}{0!5!}a^5 + \frac{5!}{1!4!}a^4b + \frac{5!}{2!3!}a^3b^2 + \frac{5!}{3!2!}a^2b^3 + \frac{5!}{4!1!}ab^4 + \frac{5!}{5!0!}b^5$$

Now each coefficient is of the form $\dfrac{5!}{(\text{power of } b)!(\text{power of } a)!}$. This holds in general and we call this type of fraction a **binomial coefficient**.

Definition 7.5. The Binomial Coefficient: For nonnegative integers n and k with $n \ge k$, the binomial coefficient $\binom{n}{k}$, read as 'n choose k', is defined by $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.[5]

Although their definition involves a fraction, the binomial coefficients are always integers.

Example 7.3.6. Evaluate the following binomial coefficients.

1. $\binom{6}{0}$ 2. $\binom{10}{7}$ 3. $\binom{4}{4}$ 4. $\binom{7}{3}$ 5. $\binom{7}{4}$

[5] You may see the notation ${}_nC_k$ in place of $\binom{n}{k}$.

Solution.

1. $\binom{6}{0} = \frac{6!}{0!(6-0)!}$
$= \frac{6!}{1 \times 6!}$
$= 1$

2. $\binom{10}{7} = \frac{10!}{7!(10-7)!}$
$= \frac{10 \times 9 \times 8 \times 7!}{7!3!}$
$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$
$= 120$

3. $\binom{4}{4} = \frac{4!}{4!(4-4)!}$
$= \frac{4!}{4!0!}$
$= \frac{4!}{4!(1)}$
$= 1$

4. $\binom{7}{3} = \frac{7!}{3!(7-3)!}$
$= \frac{7!}{3!4!}$
$= \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!}$
$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}$
$= 35$

5. $\binom{7}{4} = \frac{7!}{4!(7-4)!}$
$= \frac{7!}{4!3!}$
$= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1}$
$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}$
$= 35$

□

We return to the expansion of $(a+b)^5$, noting that $\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{5!}{0!5!}$, $\binom{5}{1} = \frac{5!}{1!(5-1)!} = \frac{5!}{1!4!}$, and so forth. This gives us

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$$

The name 'binomial coefficient' now makes more sense. Additionally, if we think of the coefficient $\binom{5}{2}$ as the number of ways we can choose the 2 b's occurring in a^3b^2 out of the 5 possible b's in $(a+b)^5$, the phrase '5 choose 2' makes better sense. We are now ready to state our last result.

Theorem 7.5. The Binomial Theorem: For any positive integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

or $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$

While we have not proved this result, it follows from pattern recognition in the expansion of $(a+b)^5$.

Notice that we have not included $\binom{n}{0}$ and $\binom{n}{n}$ in the second equation in **Theorem 7.5**, since both are equal to 1.

Example 7.3.7. Use the Binomial Theorem to expand $(a+b)^6$.

Solution. By the Binomial Theorem, we have

$$\begin{aligned}(a+b)^6 &= \sum_{k=0}^{6}\binom{6}{k}a^{6-k}b^k \\ &= a^6+\binom{6}{1}a^{6-1}b+\binom{6}{2}a^{6-2}b^2+\binom{6}{3}a^{6-3}b^3+\binom{6}{4}a^{6-4}b^4+\binom{6}{5}a^{6-5}b^5+b^6 \\ &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\end{aligned}$$

□

Example 7.3.8. Use the Binomial Theorem to expand $(2x-3)^4$.

Solution. We begin by writing $(2x-3)^4=\left((2x)+(-3)\right)^4$. Then, by the Binomial Theorem, we have

$$\begin{aligned}(2x-3)^4 &= \left((2x)+(-3)\right)^4 \\ &= \sum_{k=0}^{4}\binom{4}{k}(2x)^{4-k}(-3)^k \\ &= (2x)^4+\binom{4}{1}(2x)^{4-1}(-3)^1+\binom{4}{2}(2x)^{4-2}(-3)^2+\binom{4}{3}(2x)^{4-3}(-3)^3+(-3)^4 \\ &= 16x^4+(4)(8x^3)(-3)+(6)(4x^2)(9)+(4)(2x)(-27)+81 \\ &= 16x^4-96x^3+216x^2-216x+81\end{aligned}$$

□

The expansions in the previous two examples first showed up in **Example 7.3.3**, and referring back to that example we observe that the Binomial Theorem does not appear to be a time saver. However, the following examples demonstrate the usefulness of the Binomial Theorem.

Example 7.3.9. Find the seventh term in the expansion of $(a+b)^{10}$.

Solution. By the Binonial Theorem, the expansion of $(a+b)^{10}$ is

$$(a+b)^{10}=\sum_{k=0}^{10}\binom{10}{k}a^{10-k}b^k$$

The seventh term corresponds to $k=6$, so the seventh term is

$$\binom{10}{6}a^{10-6}b^6 = \frac{10!}{6!(10-6)!}a^{10-6}b^6 = \frac{(6!)(7\times 8\times 9\times 10)}{(6!)(1\times 2\times 3\times 4)}a^4b^6$$

After simplifying, we find the seventh term is $210a^4b^6$.

□

Example 7.3.10. Consider $(2x-y)^7$. Find the term in its expansion that contains x^3 and the term that contains y^5.

Solution. By the Binomial Theorem,

$$(2x-y)^7 = \sum_{k=0}^{7}\binom{7}{k}(2x)^{7-k}(-y)^k$$

The term that contains x^3 corresponds to $k=4$ since we want $7-k=3$. The term containing x^3 is

$$\binom{7}{4}(2x)^{7-4}(-y)^4 = \frac{7!}{4!(7-4)!}(2x)^3y^4 = \frac{5\times 6\times 7}{1\times 2\times 3}(8x^3)y^4$$

After simplifying, we find the term containing x^3 is $280x^3y^4$. Since the term that contains y^5 corresponds to $k=5$, it is

$$\binom{7}{5}(2x)^{7-5}(-y)^5 = \frac{7!}{5!(7-5)!}(2x)^2(-y)^5 = -\frac{6\times 7}{1\times 2}(4x^2)y^5$$

The term containing y^5 is $-84x^2y^5$.

□

7.3 Exercises

1. What is a binomial coefficient and how is it calculated?

2. When is it an advantage to use the Binomial Theorem? Explain.

In Exercises 3 – 8, evaluate the expression.

3. $6!$

4. $\frac{10!}{7!}$

5. $\binom{12}{6}!$

6. $\frac{100!}{99!}$

7. $\frac{7!}{2^3 3!}$

8. $\frac{9!}{4!3!2!}$

In Exercises 9 – 17, evaluate the binomial coefficient.

9. $\binom{6}{2}$

10. $\binom{5}{3}$

11. $\binom{7}{4}$

12. $\binom{8}{3}$

13. $\binom{9}{7}$

14. $\binom{10}{9}$

15. $\binom{117}{0}$

16. $\binom{25}{11}$

17. $\binom{200}{199}$

In Exercises 18 – 29, expand the given binomial.

18. $(4a-b)^3$

19. $(x+2)^5$

20. $(5a+2)^3$

21. $(3a+2b)^3$

22. $(2x+3y)^4$

23. $(2x-1)^4$

24. $(4x+2y)^5$

25. $(3x-2y)^4$

26. $(4x-3y)^5$

27. $\left(\frac{1}{3}x+y^2\right)^3$

28. $\left(\frac{1}{x}+3y\right)^5$

29. $\left(x-x^{-1}\right)^4$

In Exercises 30 – 37, use the Binomial Theorem to find the indicated term.

30. The fourth term of $(2x-3y)^4$

31. The fourth term of $(3x-2y)^5$

32. The third term of $(6x-3y)^7$

33. The eighth term of $(7+5y)^{14}$

34. The seventh term of $(a+b)^{11}$

35. The fifth term of $(x-y)^7$

36. The term containing x^3 in the expansion $(2x-y)^5$

37. The term containing x^{117} in the expansion $(x+2)^{118}$

38. You've just won three tickets to see the new film, '$8.\overline{9}$.' Five of your friends, Brenda, Cindy, Michael, Rachel and Sadie, are interested in seeing it with you. With the help of your classmates, list all the possible ways to distribute your two extra tickets among your five friends. Now suppose you've come down with the flu. List all the different ways you can distribute the three tickets among these five friends. How does this compare with the first list you made? What does this have to do with the fact that $\binom{5}{2}=\binom{5}{3}$?

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